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Maxwell Distribution of Molecular Velocities

by David Forfar

Maxwell derived his velocity distribution in two lines from the functional equation

$$f(x)f(y)f(z) = \phi(x^2 + y^2 + z^2)$$
$$\text{giving } f(x) = Ae^{-Bx^2}$$

e.g. the velocity in the x or y or z directions follows a normal distribution.

It is said that Maxwell had read Sir John Herschel's review in the Edinburgh Review of July 1850 of "Quetelet on Probabilities", which states:-

> *"Suppose a ball is dropped from a great height given the intention that it should fall on a given mark. Fall as it may, its deviation from the mark is error and the probability of that error is the unknown function of its square i.e. the sum of the deviations in any two rectangular directions. Now, the probability of any deviation depending solely on its magnitude, and not on its direction, it follows that the probability of each of these rectangular deviations must be the same function of its square. And since the observed oblique deviation is equivalent to the two rectangular ones, supposed concurrent, and which are essentially independent of one another,* and is, therefore, a compound event of which they are the simple independent constituents, therefore the probability will be the product of their separate probabilities. The form of the unknown function comes to be determined from this condition, viz., that the product of such functions of two independent elements is equal to the same function of their sum. But it is shown in every work of algebra that this property is the peculiar characteristic of, and belongs only to, the exponential function. This, then, is the function of the square of the error, which expresses the probability of committing that error."*

* That is, *the decrease or diminution in one of which may take place without increasing or diminishing the other.* On this the whole of the proof depends (Sir John Herschel, 1857).

The above analysis gives $\phi(x^2 + y^2) = \phi(x^2)\phi(y^2)$ and $\phi(x^2) = e^{-ax^2}$ or the normal distribution.

All Maxwell had to do was to translate distances into velocities and two-dimensions to three-dimensions - easy if you have genius (and everyone else kicking themselves for not having seen it!).