James Clerk Maxwell's 1871 Paper to the London Mathematical Society where he suggests the word 'curl' for a vector quantity (Proceedings of the London Mathematical Society, s1-3, 224-233)

March 9th, 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. C. R. Hodgson, B.A. Lond., was elected a Member;

and the following gentlemen were proposed for election:-

The Hon. J. W. Strutt;

Major Frederick Close, R.A.; and

Mr. James Stuart, Fellow and Tutor of Trinity College, Cambridge.

Two Models were exhibited by Professor Henrici, the one being the cast ordered for the Society at the February meeting, the other a model of the surface described at the meeting held November, 1869; it was agreed that Professor Henrici should order a copy of the latter model also to be made for the Society.

Professor H. J. S. Smith read a paper on *"Skew Cubics*," upon which Professor Cayley made a few remarks.

Mr. Tucker (Hon. Sec.) then read the following communication:-

Remarks on the Mathematical Classification of Phys ical Quantities

by J. CLERK MAXWELL, LL.D, F.R.S.

The first part of the growth of a physical science consists in the discovery of a system of quantities on which its phenomena may be conceived to depend. The next stage is the discovery of the mathematical form of the relations between these quantities. After this, the science may be treated as a mathematical science, and the verification of the laws is effected by a theoretical investigation of the conditions under which certain quantities can be most accurately measured, followed by an experimental realisation of these conditions, and actual measurement of the quantities.

It is only through the progress of science in recent times that we have become acquainted with so large a number of physical quantities that a classification of them is desirable.

One very obvious classification of quantities is founded on that of the sciences in which they occur. Thus temperature, pressure, density, specific heat, latent heat, &c., are quantities occurring in the theory of the action of heat on bodies.

But the classification which I now refer to is founded on the mathematical or formal analogy of the different quantities, and not on the matter to which they belong. Thus a finite straight line, a force, a velocity of rotation, &c., are quantities, differing in their physical nature, but agreeing in their mathematical form. We may distinguish the two methods of classification by calling the first a physical, and the second a mathematical classification of quantities.

A knowledge of the mathematical classification of quantities is of great use both to the original investigator and to the ordinary student of the science. The most obvious case is that in which we learn that a certain system of quantities in a new science stand to one another in the same mathematical relations as a certain other system in an old science, which has already been reduced to a mathematical form, and its problems solved by mathematicians.

Thus, when Mossotti observed that certain quantities relating to electro-static induction in dielectrics had been shown by Faraday to be analogous to certain quantities relating to magnetic induction in iron and other bodies, he was enabled to make use of the mathematical investigation of Poisson relative to magnetic induction, merely translating it from the magnetic language into the electric, and from French into Italian.

Another example, by no means so obvious, is that which was originally pointed out by Sir William Thomson, of the analogy between problems in attractions and problems in the steady conduction of heat, by the use of which we are able to make use of many of the results of Fourier for heat in explaining electrical distributions, and of all the results of Poisson in electricity in explaining problems in heat.

But it is evident that all analogies of this kind depend on principles of a more fundamental nature; and that, if we had a true mathematical classification of quantities, we should be able at once to detect the analogy between any system of quantities presented to us and other systems of quantities in known sciences, so that we should lose no time in availing ourselves of the mathematical labours of those who had already solved problems essentially the same.

All quantities may be classed together in one respect, that they may be defined by means of two factors, the first of which is a numerical quantity, and the second is a standard quantity of the same kind with that to be defined.

Thus number may be said to rule the whole world of quantity, and the four rules of arithmetic may be regarded as the complete equipment of the mathematician.

Position and form, which were formerly supposed to be in the exclusive possession of geometers, were reduced by Descartes to submit to the rules of arithmetic by means of that ingenious scaffolding of coordinate axes which he made the basis of his operations.

Since this great step was taken in mathematics, all quantities have been treated in the same way, and presented to the mind by means of numbers, or symbols which denote numbers, so that as soon as any science has been thoroughly reduced to the mathematical form, the solution of problems in that science, as a mental process, is supposed (at least by the outer world) to be carried on without the aid of any of the physical ideas of the science.

I need not say that this is not true, and that mathematicians, in solving physical problems, are very much aided by a knowledge of the science in which the problems occur.

At the same tune, I think that the progress of science, both in the way of discovery, and in the way of diffusion, would be greatly aided if more attention were paid in a direct way to the classification of quantities.

A most important distinction was drawn by Hamilton when he divided the quantities with which he had to do into *Scalar* quantities, which are completely represented by one numerical quantity, and *Vectors*, which require three numerical quantities to define them.

The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared, for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science.

We may imagine another step in the advancement of science to be the invention of a method, equally appropriate, of conceiving dynamical quantities. As our conceptions of physical science are rendered more vivid by substituting for the mere numerical ideas of Cartesian mathematics the geometrical ideas of Hamiltonian mathematics, so in the higher sciences the ideas might receive a still higher development if they could be expressed in language as appropriate to dynamics as Hamilton's is to geometry.

Another advantage of such a classification is, that it guides us in the use of the four roles of arithmetic. We know that we must not apply the rules of addition or subtraction unless the quantities are of the same kind. In certain cases we may multiply or divide one quantity by another, but in other cases the result of the process is of no intellectual value.

It has been pointed out by Professor Rankine that the physical quantity called Energy or Work can be conceived as the product of two factors in many different ways.

The dimensions of this quantity are $\frac{ML^2}{T^2}$, where L, M, and T represent the concrete units of length, time and mass. If we divide the energy into two factors, one of which contains L^2, both factors will be scalars. If, on the other hand, both factors contain *L*, they will be both vectors. The energy itself is always a scalar quantity.

Thus, if we take the mass and the square of the velocity as the factors, as is done in the ordinary definitions of vis-viva or kinetic energy, both factors are scalar, though one of them, the square of the velocity, has no distinct physical meaning.

Another division into apparently scalar factors is that into volume and hydrostatic pressure, though here we must consider the volume, not in itself, hut as a quantity subject to increase and diminution, and this change of volume can only occur at the surface, and is due to a variation of the surface in the direction of the normal, so that it is not a scalar but a vector quantity. The pressure also, though the abstract conception of a hydrostatic pressure is scalar, must be conceived as applied at a surface, and thus becomes a directed quantity or vector.

The division of the energy into vector factors affords results always capable of satisfactory interpretation. Of the two factors, one is conceived as a tendency towards a certain change, and the other as that change itself.

Thus the elementary definition of *Work* regards it as the product of a force into the distance through which its point of application moves, resolved in the direction of the force. In the language of Quaternions, it is the scalar part of the product of the force and the displacement.

These two vectors, the force and the displacement, may be regarded as types of many other pairs of vectors, the products of which have for their scalar part some form of energy.

Thus, instead of dividing kinetic energy into the factors *"mass"* and *"square of velocity"*, the latter of which has no meaning, we may divide it into *"momentum"* and *"velocity"*, two vectors which, in the dynamics of a particle, are in the same direction, but, in generalized dynamics, may be in different directions, so that in taking their product we must remember the rule for finding the scalar part of it.

But it is when we have to deal with continuous bodies, and quantities distributed in space, that the general principle of the division of energy into two factors is most clearly seen.

When we regard energy as residing intrinsically in a body, we may measure its intensity by the amount contained in unit of volume. This is, of course, a scalar quantity.

Of the factors which compose it, one is referred to unit of length, and the other to unit of area. This gives what I regard as a very important distinction among vector quantities.

Vectors which are referred to unit of length I shall call Forces, using the word in a somewhat generalized sense, as we shall see. The operation of taking the integral of the resolved part of a force in the direction of a line for every element of that line, has always a physical meaning. In certain cases the result of the integration is independent of the path of the line between its extremities. The result is then called a Potential.

Vectors which are referred to unit of area I shall call Fluxes. The operation of taking the integral of the resolved part of a flux perpendicular to a surface for every element of the surface has always a physical meaning. In certain cases the result of the integration over a closed surface is independent, within certain restrictions, of the position of the surface. The result then expresses the Quantity of some kind of matter, either existing within the surface, or flowing out of it, according to the physical nature of the flux.

In many physical cases, the force and the flux are always in the same direction, and proportional to each other. The one is therefore used as the measure of the other, their symbols degenerate into one, and their ideas become confounded together. One of the most important mathematical results of the discovery of substances having different physical properties in different directions, has been to enable us to distinguish between the force and the flux, by letting us see that their directions may be different.

Thus, in the ordinary theory of fluids, in which the only motion considered is that which we can directly perceive, we may define the velocity equally well in two different ways. We may define it with reference to unit of length, as the number of such units described by a particle in unit of time; or we may define it with reference to unit of area, as the volume of the fluid which passes through unit of area in unit of time. If defined in the first way, it belongs to the category of forces ; if denned in the second way, to the category of fluxes.

But if we endeavour to develop a more complete theory of fluids, which shall take into account the facts of diffusion, where one fluid has a different velocity from another in the same place; or if we accept the doctrine, that the molecules of a fluid, in virtue of the heat of the substance, are in a state of agitation ; then, though we may give a definition of the velocity of a single molecule with reference to unit of length, we cannot do so for the fluid; and the only way we have of defining the motion of the fluid is by considering it as a flux, and measuring it by the mass of the fluid which flows through unit of area.

This distinction is still more necessary when we come to heat and electricity. The flux of heat or of electricity cannot be even thought of in any way except as the quantity which flows through a given area in a given time. To form a conception of the velocity, properly so called, of either agent, would require us to conceive heat or electricity as a continuous substance having a known density.

We must therefore consider these quantities as fluxes. The forces corresponding to them are, in the case of heat, the rate of variation of temperature, and, in the case of electricity, the rate of variation of potential.

I have said enough to point out the distinction between forces and fluxes. In statical electricity the resultant force at a point is the rate of variation of potential, and the flux is a quantity, hitherto confounded with the force, which I have called the electric displacement.

In magnetism the resultant force is also the rate of variation of the potential, and the flux is what Faraday calls the magnetic induction, and is measured, as Thomson has shown, by the force on a unit pole placed in a narrow crevasse cut perpendicular to the direction of magnetization of the magnet. I shall not detain the Society with the explanation of these quantities, but I must briefly state the nature of a ratio of a force to a flux in its moat general form.

When one vector is a function of another vector, the ratio of the first to the second is, in general, a quaternion which is a function of the second vector.

When, if the second vector varies in magnitude only, the first is always proportional to it, and remains constant in direction, we have the important case of the function being a *linear* one. The first vector is then said to be a linear and vector function of the second.

If α, β, χ are the Cartesian components of the first vector, and *a, b, c* those of the second, then:-

$$\alpha = r_1.a + q_3.b + p_2.c,$$
$$\beta = p_3.a + r_2.b + q_1.c,$$
$$\chi = q_2.a + p_1.b + r_3.c,$$

where the coefficients *p*, *q*, *r* are constants. When the *p's* are equal to the corresponding *q's*, the function is said to be self-conjugate. It may then be represented geometrically as the relation between the radius vector from the centre of an ellipsoid, and the perpendicular on the tangent plane.

We may remark that even here, where we may seem to have reached a purer air, uncontaminated by physical applications, one vector is essentially a line, while the other is defined as the normal to a plane, as in all the other pairs of vectors already mentioned.[1]

Another distinction among physical vectors is founded on a different principle, and divides them into those which are defined with reference to translation and those which are defined with reference to rotation. The remarkable analogies between these two classes of vectors is well pointed out by Poinsôt in his treatise on the motion of a rigid body. But the most remarkable illustration of them is derived from the two different ways in which it is possible to contemplate the relation between electricity and magnetism.

Helmholtz, in his great paper on Vortex Motion, has shown how to construct an analogy between electro-magnetic and hydro-kinetic phenomena, in which magnetic force is represented by the velocity of the fluid, a species of translation, while electric current is represented by the rotation of ecectro-magnetism; for though the analogy is perfect in form, the dynamics of the two systems are different.

According to Ampère and all his followers, however, electric currents are regarded as a species of translation, and magnetic force as depending on rotation. I am constrained to agree with this view, because the electric current is associated with electrolysis, and other undoubted instances of translation, while magnetism is associated with the rotation of the plane of polarization of light, which, as Thomson has shown, involves actual motion of rotation.

The Hamiltonian operator $V\nabla$, applied to any vector function, converts it from translation to rotation, or from rotation to translation, according to the kind of vector to which it is applied.

I shall conclude by proposing for the consideration of mathematicians certain phrases to express the results of the Hamiltonian operator Δ. I should be greatly obliged to anyone who can give me suggestions on this subject, as I feel that the onomastic power is very faint in me, and that it can be successfully exercised only in societies.

$$\nabla \text{ is the operation } i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

where *i*, *j*, *k* are unit vectors parallel to *x*, *y*, *z* respectively. The result of performing this operation twice on any subject is the well known operation (of Laplace)

$$\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right).$$

The discovery of the square root of this operation is due to Hamilton; but most of the applications here given, and the whole development of the theory of this operation, are due to Prof. Tait, and are given in several papers, of which the first is in the Proceedings of the Royal Society of Edinburgh, April 28, 1862,

[1] The subject of linear equations in quaternions has been developed by Professor Tait, in several communications to the Royal Society of Edinburgh.

and the most complete is that on *"Green's and other allied Theorems"*, Transactions of the Royal Society of Edinburgh, 1869-70.

And first, I propose to call the result of ∇^2 (Laplace's operation with the negative sign) the ***Concentration*** of the quantity to which it is applied.

For if Q be a quantity, either scalar or vector, which is a function of the position of a point; and if we find the integral of Q taken throughout the volume of a sphere whose radius is *r;* then, if we divide this by the volume of the sphere, we shall obtain $\bar{Q}$, the ***mean*** value of Q within the sphere. If Q_0 is the value of Q at the centre of the sphere, then, when *r* is small,

$$Q_0 - \bar{Q} = \frac{r^2}{10}.\nabla^2 Q,$$

or the value of Q at the centre of the sphere exceeds the mean value of Q within the sphere by a quantity depending on the radius, and on $\nabla^2 Q$. Since, therefore, $\nabla^2 Q$ indicates the excess of the value of Q at the centre above its mean value in the sphere, I shall call it the concentration of Q.

If Q is a scalar quantity, its concentration is, of course, also scalar. Thus, if Q is an electric potential, $\nabla^2 Q$ is the density of the matter which produces the potential.

If Q is a vector quantity, then both Q_0 and $\bar{Q}$ are vectors, and $\nabla^2 Q$ is also a vector, indicating the excess of the uniform force Q_0 applied to the whole substance of the sphere above the resultant of the actual force Q acting on all the parts of the sphere.

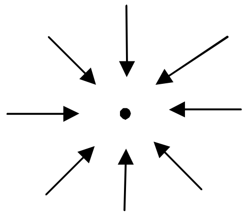
Let us next consider the Hamiltonian operator ∇. First apply it to a scalar function P. The quantity ∇P is a vector, indicating the direction in which P decreases most rapidly, and measuring the rate of that decrease. I venture, with much diffidence, to call this the *slope* P. Lamé calls the *magnitude* of ∇P the *"first differential parameter"* of P, but its *direction* does not enter into his conception. We require a vector word, which shall indicate both direction and magnitude, and one not already employed in another mathematical sense. I have taken the liberty of extending the ordinary sense of the word *slope* from topography, where only two independent variables are used, to space of three dimensions.

If σ represents a vector function, $\nabla\sigma$ may contain both a scalar and a vector part, which may be written $S\nabla\sigma$ and $V\nabla\sigma$.

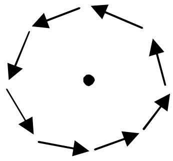
I propose to call the scalar part the *Convergence* of u, because, if a closed surface be described about any point, the surface integral of σ, which expresses the effect of the vector σ considered as an inward flux through the surface, is equal to the volume integral of $S\nabla\sigma$ throughout the enclosed space. I think, therefore, that the *convergence* of a vector function is a very good name for the effect of that vector function in carrying its subject inwards towards a point.

But $\nabla\sigma$ has, in general, also a vector portion, and I propose, but with great diffidence, to call this vector the ***Curl*** or ***Version*** of the original vector function.

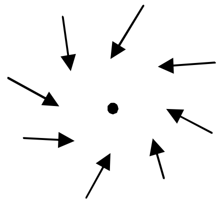
It represents the direction and magnitude of the rotation of the subject matter carried by the vector σ. *I* have sought for a word which shall neither, like ***Rotation, Whirl,*** or ***Twirl***, connote motion, nor, like ***Twist***, indicate a helical or screw structure which is not of the nature of a vector at all.



CONVERGENCE



CURL



CURL AND CONVERGENCE

If we subtract from the general value of the vector function σ its value σ_0 at the point P, then the remaining vector $\sigma - \sigma_0$ will, when there is pure convergence, point towards P. When there is pure curl, it will point tangentially round P; and when there is both convergence and curl, it will point in a spiral manner.

The following statements are true:-

The slope of a scalar function has no curl.
The curl of a vector function has no convergence.
The convergence of the slope of a scalar function is its concentration.
The concentration of a vector function is the slope of its convergence, together with the curl of its curl.

The quaternion expressions, of which the above statements are a translation, were given by Prof. Tait, in his paper in the Proceedings of the Royal Society of Edinburgh, April 28th, 1862; but for the more complete mathematical treatment of the operator ∇, see a very able paper by Prof. Tait, *"On Green's and other Allied Theorems"* (Transactions of the Royal Society of Edinburgh, 1870) and another paper in the Proceedings of the Royal Society of Edinburgh. for 1870-71, p. 318.