The following Smith's Prize Examination was set by James Clerk Maxwell in 1879 when he was Cavendish Professor at the University of Cambridge. Maxwell died in November of that year.

***Historical Note*:-** Robert Smith (1689-1768) after whom the Prizes are named, was Plumian Professor of Astronomy from 1716-1760 and Master of Trinity from 1742. The Smith's Prizes (two) of 1879 were shared equally by M.J.M. Hill (5^{th} Wrangler) who became the Professor of Pure Mathematics at University College, London and A.J. Wallis (6^{th} Wrangler) who became a Fellow and Tutor at Corpus Christi College Cambridge although A.J. C. Allen was the Senior Wrangler. The exam was also sat by Karl Pearson, FRS (3^{rd} Wrangler) who also became a professor at University College, London and who describes taking the examination in Maxwell's dining room in an article entitled *Old Tripos Days at Cambridge, as seen from another viewpoint* (Mathematical Gazette, **20** (1936)) and who did best on Maxwell's paper. There is evidence from what Karl Pearson says that, as Maxwell was normally thoughtful, considerate and kind, Maxwell was already suffering from the stomach cancer that was to kill him a few months later. Karl Pearson coined the phrase 'standard deviation' in 1893.

Maxwell, who was a former pupil of the Edinburgh Academy**,** himself sat for the Smith's Prizes in 1854 being declared equal first with Routh. The 1854 question paper contains as a question in vector analysis which has now become known as Stokes' Theorem (although it was first proved by Lord Kelvin and communicated to Stokes by Kelvin in a letter of 2 July 1850 and should be known by Kelvin's name). Since Maxwell said later *"It* (i.e. the result) *was not entirely new to yours truly"* we can suppose that Maxwell solved it in the exam room. Maxwell was not to know then the vital role which Stokes' Theorem was to play in enabling him to 'mathematise' Faraday.

The James Clerk Maxwell Foundation is grateful to :-

Dr. Jovan Jevtic is Assistant Professor at the Milwaukee School of Engineering, USA who solved many of the problems - his webpage is at Dr. Jovan Jevtic

Raúl A. Simón is from Santiago, Chile (see Questions 1 and 6)

Diego Sevilla is from Santa Fe, Argentina (see Question 12)

Winifred Sillitto, is former member Applied Optics Group, University of Edinburgh, webpage at Winifred Sillitto

Douglas Essex is at College of Optical Sciences, University of Arizona, U.S.A. webpage College of Optical Sciences (see Question 9)

J. Stewart Fowlie is ex. Christ's College Cambridge, and former Mathematics Master at Edinburgh Academy has made contributions to the Mathematical Gazette, see, for example, The Mathematical Association - supporting mathematics in education

Martin Baxter is ex. Pembroke College, Cambridge and former pupil of Daniel Stewart's College, Edinburgh webpage at Martin Baxter (see Question 3)

David Forfar is a Trustee of the above Foundation, ex. Trinity, Cambridge and former pupil of Edinburgh Academy, webpage David O Forfar

Comments on the Fluid Dynamics (questions 10 and 12) have been given by **Professor H.K. Moffatt, FRS, FRSE** the Professor of Fluid Dynamics at the University of Cambridge and former Director of the Isaac Newton Institute and former pupil of George Watson's College Edinburgh, webpage Keith Moffatt

WEDNESDAY, *January* 29, 1879

By J. CLERK MAXWELL, PROFESSOR OF EXPERIMENTAL PHYSICS.
(Trinity College, Cambridge)

Question 1: IF in a plane *AB* = *CD*, and if *P* be the intersection of the lines which bisect *AC* and *BD* perpendicularly, *Q* that of the lines which bisect *AD* and *BC* perpendicularly, and *R* the intersection of *AB* and *CD*, show that *PR* and *QR* are the interior and exterior bisectors of the angle *ARC*.

Solution by David Forfar

The triangles PAB and PCD have three equal sides and are therefore congruent. The quadrilateral PRBD is cyclic as PR subtends equal angles at the circumference. The angle PRD is therefore equal to PBD. The complement of the angle PRB (i.e. 180° minus the angle) is equal to the angle RBP plus the angle RPB and the latter is equal to angle RDB and using the congruent triangles we have angle RBP equal to RPD.
Thus the complement of the angle PRB is equal to the angle PDB and because of isosceles triangles we have angle PBD equal to PDB. The other case is similar.

Solution by Raúl Simón

The triangles PAB and PCD have three equal sides and are therefore congruent. Drop perpendiculars from P onto the two lines BAR and DCR and let the foot of these perpendiculars be E and F respectively. The triangles PED and PFD are congruent (two angles and the enclosed side) so PE and PF are of equal length. Thus P is on the internal bisector of angle DRE. In the other case the triangles QAB and QCD are congruent therefore the perpendiculars from Q onto the two lines BAR and DCR are of equal length so that Q bisects the angle ARD.

Question 2. Three points *A*, *B*, *C* on a straight line correspond homographically to three points *a*, *b*, *c* on another straight line: give a geometrical construction to determine the point on either line which corresponds to a point at an infinite distance on the other.
Four points *A*, *B*, *C*, *D*, of which no three are in a straight line, correspond homographically to four points *a*, *b*, *c*, *d*: prove the following construction for finding the equiangular foci of the two figures, at which corresponding lines subtend equal angles.—Let *Y*, *Z*, *y*, *z* be the vanishing points on the lines *AC*, *AB*, *ac*, *ab* respectively; on *YZ* describe the triangle ZF_1Y similar to *zay*, and on *yz* describe the triangle zf_1y similar to *ZAY*, then F_1 and f_1 are the positive equiangular foci, and their images in *YZ* and *yz* respectively are the negative foci.

Solution by Jovan Jevtic

Construction. To find the vanishing point on the line ABC: translate the line abc to $a'b'c'$ such that the point c' coincides with C; let O be the point of intersection of the lines Aa' and Bb'; then a parallel to the line abc drawn through the point O intersects the line ABC at the desired vanishing point.

Proof. Let X be the point on the line ABC as constructed and let i be a point at an infinite distance on the line abc. To prove that the point X corresponds homographically to the point i, it suffices to prove the equality of the cross ratios of the points i,a,b,c and X,A,B,C, because the cross-ratio of four collinear points is an invariant of homography.

Consider the pencil of rays Oi', Oa', Ob', Oc' where i' is a point at an infinite distance on the line $a'b'c'$. The line ABC intersects the rays at the points $X, A, B,$ and C, respectively, and the equality of the cross-ratios on the common pencil of rays gives:

$$(X, A, B, C) = \frac{XA \cdot BC}{XC \cdot AB} = \lim_{|i'a'| \to \infty} \frac{i'a' \cdot b'c'}{i'c' \cdot a'b'} = \frac{b'c'}{a'b'} \tag{2.1}$$

On the other hand, the cross-ratio of the points i, a, b, c is equal to:

$$(i, a, b, c) = \lim_{|ia| \to \infty} \frac{ia \cdot bc}{ic \cdot ab} = \frac{bc}{ab} \tag{2.2}$$

But $b'c' = bc$ and $a'b' = ab$ by construction, proving the equality of the cross-ratios (2.1) and (2.2) and the homographic correspondence of the points X and i.

Key Proportions. Consider an arbitrary point Q and its corresponding point q. Let X and x be the vanishing points on the lines AQ and aq, respectively. Let I and i be the points at an infinite distance on the lines AQ and aq, respectively. Then the points X, A, Q, I correspond homographically to the points i, a, q, x, implying the equality of the cross-ratios:

$$\lim_{|AI| \to \infty} \frac{XA \cdot QI}{XI \cdot AQ} = \lim_{|ai| \to \infty} \frac{ia \cdot qx}{ix \cdot aq} \qquad \to \qquad \frac{XA}{AQ} = \frac{qx}{aq} \tag{2.3}$$

Furthermore, the vanishing points X and x must lie on the straight lines YZ and yz, respectively, because the locus of the vanishing points is a straight line. Let W and w be the points at an infinite distance on the lines XYZ and xyz. It is easy to show that w corresponds homographically to W. Consider the corresponding pencils of rays AW, AQ, AC, AB and aw, aq, ac, ab. Since the collinear points W, X, Y, Z and w, x, y, z lie on the corresponding pencils, their cross-ratios are equal:

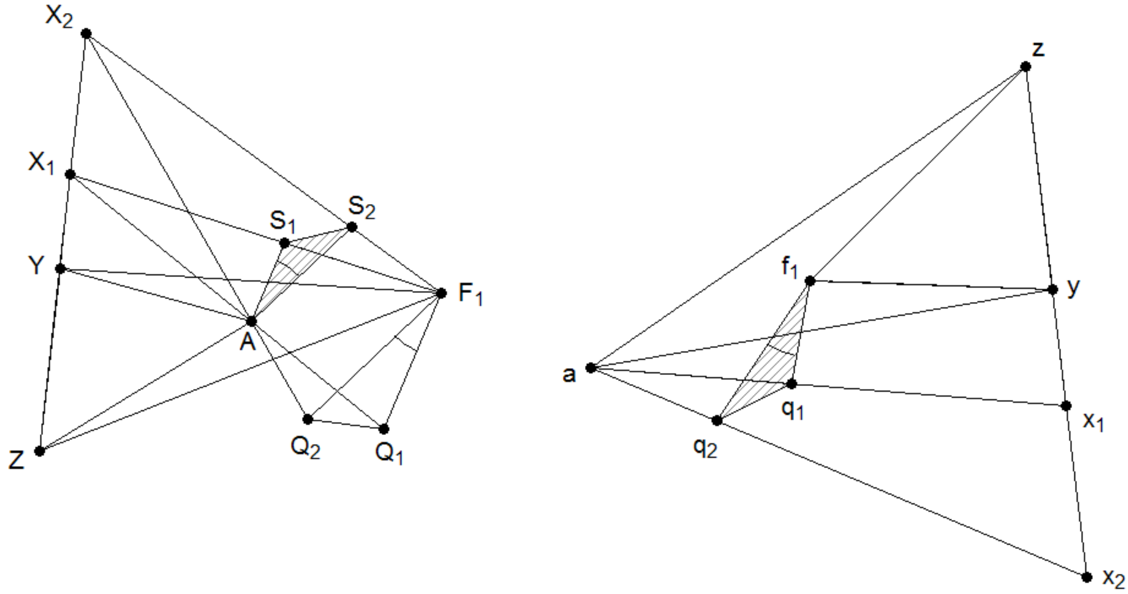
$$\lim_{|ZW| \to \infty} \frac{WX \cdot YZ}{WZ \cdot XY} = \lim_{|zw| \to \infty} \frac{wx \cdot yz}{wz \cdot xy} \qquad \to \qquad \frac{YZ}{XY} = \frac{yz}{xy} \tag{2.4}$$

Proof for the Positive Foci. Let Q_1 and Q_2 be two arbitrary points <u>on the same side</u> of the vanishing line XYZ and let q_1 and q_2 be the corresponding points. Let X_1, X_2, x_1, x_2 be the vanishing points on the lines AQ_1, AQ_2, aq_1, aq_2, respectively. The figures $YZAF_1$ and yzf_1a are similar by construction. In addition, from (2.4) we conclude that the segments defined by Z, Y, X_1, X_2 are similar to the segments defined by z, y, x_1, x_2. Consequently, the figures $YZAF_1X_1X_2$ and $yzf_1ax_1x_2$ are similar.

Let S_1 be the point on F_1X_1 such that the line AS_1 is parallel to the line Q_1F_1. Then the triangles $Q_1X_1F_1$ and AX_1S_1 are similar and we have:

$$\frac{X_1S_1}{S_1F_1} = \frac{X_1A}{AQ_1} = \frac{q_1x_1}{aq_1} \tag{2.5}$$

where the second equality is due to (2.3). From (2.5) we see that the point S_1 in the figure $YZAF_1X_1X_2$ divides the segment F_1X_1 in the same proportion in which the point q_1 in the similar figure $yzf_1ax_1x_2$ divides the segment ax_1. Consequently, the similarity extends to the figures $YZAF_1X_1X_2S_1$ and $yzf_1ax_1x_2q_1$.



In a similar manner, let S_2 be the point on F_1X_2 such that the line AS_2 is parallel to the line Q_2F_1. Following the same line of reasoning we conclude that the similarity extends to the figures $YZAF_1X_1X_2S_1S_2$ and $yzf_1ax_1x_2q_1q_2$.

Consider the triangles S_1AS_2 and $q_1f_1q_2$ whose similarity has just been demonstrated. We have:

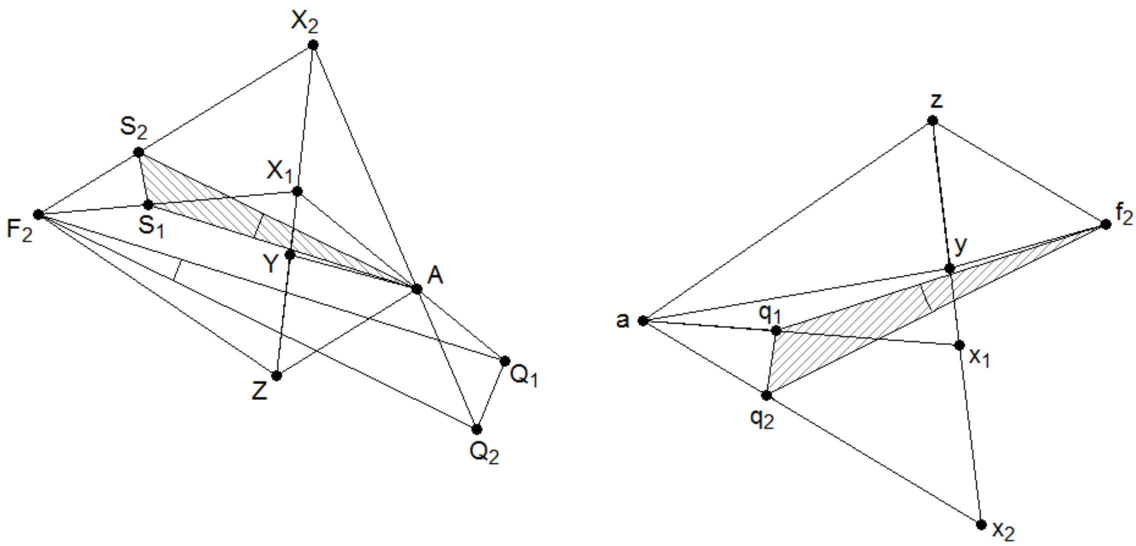
$$\boxed{angle\ q_1f_1q_2 = angle\ S_1AS_2 = angle\ Q_1F_1Q_2} \tag{2.6}$$

where the first equality is due to the similarity of the triangles (shown shaded) and the second equality is due to the parallelism of the lines AS_1 and AS_2 to the lines Q_1F_1 and Q_2F_1, respectively. The result (2.6) shows that the arbitrary corresponding segments Q_1Q_2 and q_1q_2 subtend equal angles at the points F_1 and f_1, respectively, thus proving that the points F_1 and f_1 are indeed the positive equiangular foci.

Proof for the Negative Foci. If F_2 and f_2 are substituted for F_1 and f_1, respectively, the proof proceeds in exactly the same way as the proof for the positive foci, except for the opposite signs of the subtended angles:

$$\boxed{angle\ q_1f_2q_2 = angle\ Q_2F_1Q_1 = -angle\ Q_1F_2Q_2} \tag{2.7}$$

This proves that the points F_2 and f_2 are indeed the negative equiangular foci.



Remark. The results (2.6) and (2.7) hold when the points Q_1 and Q_2 are on the same side of the vanishing line XYZ. If the points Q_1 and Q_2 lie <u>on the opposite sides</u> of the vanishing line, we submit without a proof that the results are trivially modified as follows:

$$\begin{aligned} &angle\ q_1 f_1 q_2 = angle\ Q_1 F_1 Q_2 \pm 180^o \\ &angle\ q_1 f_2 q_2 = -angle\ Q_1 F_2 Q_2 \pm 180^o \end{aligned} \tag{2.8}$$

Question 3. Show that if two points are taken at random in a region of volume *V*, the probability of their distance being between *a* and *a*+*da*, where *a* is small compared with the dimensions of the region, is:-

$$4\pi V^{-1} a^2 da.$$

Show also that if three and four points are taken at random, the probabilities of their being within specified limits of distance are:-

$$8\pi^2 V^{-2} abcdadbdc \text{ and } \frac{8}{3}\pi^2 V^{-3} T^{-1} abclmndadbdcdldmdn$$

respectively, where *a*, *b*, *c*, *l*, *m*, *n* are the distances between the points, and *T* is the volume of the tetrahedron whose angles are the four points.

<u>Solution by Martin Baxter</u>

Assume that the region has most of its volume away from its edges (so that a randomly chosen point is very likely to be more than distance *a* away from its boundary) and also that in each of the problems, there do exist points of the given distances away from each other (for example, for three points, there is a triangle of sides *a*, *b* and *c*).

We proceed recursively, if we have *k* points correctly aligned, we calculate the conditional probability that an additional point is the correct distances, $a_1, a_2, \ldots, a_k$ away from each of those k points. This boils down to the new point being in the set formed by the intersection of spherical shells of the form:

$$A_k = \{x \in \mathbf{R}^k : | x - x_i | \in \left[a_{i,} a_i + da_i \right], i = 1,....k\}, \qquad (3.1)$$

where $x_1, \ldots x_k$ are the existing k points. The probability of a randomly chosen point being in this set is just the set's volume divided by V, the volume of the region.

(1) For two points, the set A_1 is simply a spherical shell with volume equal to its surface area ($4\pi a^2$) multiplied by the shell's thickness (da). This gives the desired probability when normalised by V. It is useful to prove the following Lemma.

Lemma. *If $y_1, \ldots, y_k$ are a set of spanning vectors in $\mathbf{R}^k$, then the volume of the set*

$$S = \{x \in \mathbf{R}^k : 0 \le x^{\mathrm{T}} y_i \le 1,\ i = 1,...k\} \qquad (3.2)$$

is the reciprocal of the volume of the k-dimensional parallelepiped, P, whose sides are formed by the vectors $y_1, \ldots, y_k$.

Proof. Consider f and g, two linear maps from $\mathbf{R}^k$ to itself, where

$$f : x \mapsto (x^{\mathrm{T}} y_i)_{i=1}^{i=k}, \qquad g : x \mapsto \sum_{i=1}^{i=k} x_i y_i$$

and notice that they are adjoint and so share the same Jacobian determinant, |J|. Letting T be the set $\{x \in \mathbf{R}^k : 0 \le x_i \le 1,\ i = 1,...k\}$, then we see that $S = f^{-1}(T)$ and $P = g(T)$. As T has volume of one unit, it follows that $|S| = |J|^{-1}$ and $|P| = |J|$, whence the result.

(2) For three points, we can assume that a is the largest distance and that $a < b_1 + b_2$, where we rewrite b and c as b_1 and b_2. We already have the points x_1 and x_2 chosen so that $|x_1 - x_2| = a$, and we wish the third point to lie in the intersection

$$A_2 = \{x \in \mathbf{R}^3 : | x - x_i | \in \left[b_i, b_i + db_i \right], i = 1, 2\} \qquad (3.3)$$

Let us consider first the locus of points

$$A_2^0 = \{x \in \mathbf{R}^3 : | x - x_i | \in b_i, i = 1, 2\} \qquad (3.4)$$

which form a circle around the line $x_1 : x_2$. Let us fix any point $x_3 \in A_2^0$, and let n_1, n_2 and n_3 be unit vectors in the directions $x_3 - x_1, x_3 - x_2$ and $x_2 - x_1$ respectively; and let Q be the plane containing the three points.

The set A_2 is a torus formed by rotating a small cross-section around the circle A_2^0. That cross-section is the two-dimensional region

$$\{x \in Q; 0 \le n_i^{\mathrm{T}}(x - x_3) \le db_i, i = 1,2\}, \quad (3.5)$$

which has area, by the Lemma, equal to $db_1 db_2 / \sin\theta,$, where θ is the angle between n_1 and n_2. Let φ be the angle between n_1 and n_3, so that $\sin\varphi = a^{-1} b_2 \sin\theta$, by the Sine Rule. The radius of the circle A_2^0 is $b_1 \sin\varphi$, which is equal to $a^{-1} b_1 b_2 \sin\theta$. So the volume of A_2 is equal to $2\pi a^{-1} b_1 b_2 db_1 db_2$, which gives the desired result.

(3) For four points, we proceed similarly, and have already chosen x_1, x_2 and x_3 satisfactorily, and define

$$A_3^0 = \{x \in \mathbf{R}^3 : |x - x_i| = l_i, i = 1,2,3\} \quad (3.6)$$

and

$$A_3 = \{x \in \mathbf{R}^3 : |x - x_i| = [l_i, l_i + dl_i], i = 1,2,3\} \quad (3.7)$$

Then the set A_3^0 consists of just two points, one on either side of the plane Q. Choose x_4 in A_3^0, and let n_i be the unit vectors in the directions $x_4 - x_i$, for i = 1,2,3. Then, half of A_3 is the set

$$\{x \in \mathbf{R}^3 : 0 \le n_i^{\mathrm{T}}(x - x_4) \le dl_i, 1 = 1,2,3\} \quad (3.8)$$

which, by the Lemma, has volume equal to $dl_1 dl_2 dl_3$ divided by the volume of the parallelepiped formed by n_1, n_2 and n_3. That volume is equal to $6T/(l_1 l_2 l_3)$, where T is the volume of the tetrahedron. The parallelepiped has volume equal to the triple product $n_1.(n_2 \times n_3)$, which is six times as large as the volume of a tetrahedron with vertices *0, n_1,n_2,n_3*. The answer then follows.

Question 4. Thirty rods of equal length have their ends jointed five and five together so as to form the edges of a regular icosahedron: an elastic string is stretched between two opposite angles of the figure. Show that if the tension of the string is $2\sqrt{5}\left(10 - 2\sqrt{5}\right)^{\frac{1}{2}}$, the ten rods which meet the string will each have a pressure 4; the ten rods which are perpendicular to the string will each have a tension $\sqrt{5} + 1$, and the other 10 rods will each have a pressure $\sqrt{5} - 1$.

Solution by David Forfar and J. S. Fowlie

The regular icosahedron is made up of 20 equilateral triangles and, if the string is vertical, we may call A the angle between the vertical and the 5 top sides (i.e. the rods) and B the angle which the nearly vertical sides (rods) make with an imaginary vertical plumb-line hanging at a corner.

The strange numbers in the question are because of the balance in the 4 forces (T_1, 4, T_2, T_3) acting along:-(1) the string, (2) the 5 sides (i.e. the top rods) meeting at top or bottom, (3) the 10 sides perpendicular to string, (4) the other 10 sides.

72^o is the interior angle of a regular pentagon. By drawing a rough icosahedron, you can work out that sinA=1/(2sin36^o) and sinB=1/(2sin18^o) and, by resolving forces at the top, the tension in the string T_1 must be 5*4cosA. By resolving forces vertically at a corner, the force T_3 , in the nearly vertical rods, must be such that $2T_3\cos B=4\cos A$ and by resolving forces horizontally at a corner $4\sin A+2T_3\sin B\cos 72^o=2T_2\cos 54^o$.

Solving numerically, agrees with Maxwell's solution namely (T_1,4,T_2,T_3) = (10.5146, 4, 3.2361, 1.2361) but his expression depend on $\cos 72^o=\frac{1}{4}(\sqrt{5}-1)$ and $\cos 36^o=\frac{1}{4}(\sqrt{5}+1)$. These results follow from the equation cos3θ+cos2θ=0 for which the solutions, in terms of cosθ, are $\cos 36^o, \cos 72^o, -1$ and therefore expanding we have $4c^3+2c^2-3c-1=0$ where c=cosθ which factorises as $(c+1)(4c^2-2c-1)=0$ and the factors of the quadratic $4c^2-2c-1=0$ are $c=\frac{1}{4}(\sqrt{5}\pm 1)$.

Question 5. Three grooves are cut in a horizontal table; the bottoms of the grooves are horizontal lines meeting in a point *O* at angles of 120°; the sides of the grooves are planes inclined 45° to the vertical: a tripod is placed with its feet in the grooves, each foot being distant *a* from the point *O*. Suppose one of the feet is due east of *O*, and that it is made to slide up the south side of the groove in which it stands: show that the tripod will begin to move about a right hand screw, whose pitch is $\frac{3}{5}a$, and whose axis meets the plane of the feet at a point $\frac{4}{5}a$ due west of the point *O*, and is inclined $\tan^{-1}2$ in the plane of the meridian, measured from the zenith towards the north.

Solution by Jovan Jevtic

Rotation about the origin. Let x, y, z be the rectangular axes with the origin at O, pointing eastward, northward, and upward, respectively. Let $\vec{r}_A, \vec{r}_B, \vec{r}_C$ be the initial values of the radius vectors of the tripod's feet:

$$\vec{r}_A = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{r}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a \\ \frac{\sqrt{3}}{2}a \\ 0 \end{bmatrix}, \qquad \vec{r}_C = \begin{bmatrix} x_C \\ y_C \\ z_C \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a \\ -\frac{\sqrt{3}}{2}a \\ 0 \end{bmatrix}. \tag{5.1}$$

An infinitesimal rigid body displacement of the tripod can be decomposed into a rotation $d\vec{\Omega}$ and a translation $d\vec{r}_T$:

$$d\vec{r} = d\vec{\Omega}\times\vec{r} + d\vec{r}_T\,, \tag{5.2}$$

where $\vec{r}$ is the radius vector of a tripod point. Setting $\vec{r}=\vec{r}_A$, $\vec{r}=\vec{r}_B$, and $\vec{r}=\vec{r}_C$ in (5.2):

$$\begin{aligned}
&dx_A = dx_T, && dx_B = -\tfrac{\sqrt{3}}{2}a\,d\Omega_z + dx_T, && dx_C = \tfrac{\sqrt{3}}{2}a\,d\Omega_z + dx_T,\\
&dy_A = a\,d\Omega_z + dy_T, && dy_B = -\tfrac{1}{2}a\,d\Omega_z + dy_T, && dy_C = -\tfrac{1}{2}a\,d\Omega_z + dy_T,\\
&dz_A = -a\,d\Omega_y + dz_T, && dz_B = \tfrac{\sqrt{3}}{2}a\,d\Omega_x + \tfrac{1}{2}a\,d\Omega_y + dz_T, && dz_C = -\tfrac{\sqrt{3}}{2}a\,d\Omega_x + \tfrac{1}{2}a\,d\Omega_y + dz_T,
\end{aligned} \tag{5.3}$$

where $d\Omega_x, d\Omega_y, d\Omega_z$ and dx_T, dy_T, dz_T are the rectangular components of the vectors $d\vec{\Omega}$ and $d\vec{r}_T$, respectively. Let ds be an infinitesimal southward displacement of the east foot. Then:

$$\boxed{\begin{aligned} dx_A &= 0, \\ dy_A &= -ds, \\ dz_A &= ds, \end{aligned}} \qquad \begin{aligned} \sqrt{3}dx_B + dy_B &= 0, \\ dz_B &= 0, \end{aligned} \qquad \begin{aligned} \sqrt{3}dx_C - dy_C &= 0, \\ dz_C &= 0, \end{aligned} \tag{5.4}$$

because the east foot slides up the south side of the eastward groove, and the other two feet slide along the bottoms of their respective grooves. Using (5.4) in (5.3), we obtain:

$$dx_T = 0, \tag{5.5}$$

$$-2ad\Omega_z + dy_T = 0, \tag{5.6}$$

$$ad\Omega_z + dy_T = -ds, \tag{5.7}$$

$$\tfrac{\sqrt{3}}{2}ad\Omega_x + \tfrac{1}{2}ad\Omega_y + dz_T = 0, \tag{5.8}$$

$$-ad\Omega_y + dz_T = ds, \tag{5.9}$$

$$-\tfrac{\sqrt{3}}{2}ad\Omega_x + \tfrac{1}{2}ad\Omega_y + dz_T = 0, \tag{5.10}$$

which can be solved for $d\vec{\Omega}$ and $d\vec{r}_T$ in terms of ds:

$$\boxed{d\vec{\Omega} = -\frac{ds}{3a}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}, \qquad d\vec{r}_T = \frac{ds}{3}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}. \tag{5.11}$$

Screw motion. Given an infinitesimal displacement (5.2), it is possible to choose a different center of rotation $\vec{r}_0$ such that the corresponding translation is parallel to the axis of rotation:

$$\boxed{d\vec{r} = d\vec{\Omega} \times (\vec{r} - \vec{r}_0) + pd\vec{\Omega}}, \tag{5.12}$$

which then represents a motion about a right hand screw of pitch p. Comparing (5.12) to (5.2), we find:

$$-d\vec{\Omega} \times \vec{r}_0 + pd\vec{\Omega} = d\vec{r}_T. \tag{5.13}$$

A dot product with $d\vec{\Omega}$ gives $pd\Omega^2 = d\vec{r}_T \cdot d\vec{\Omega}$, which may be evaluated using (5.11):

$$\boxed{p = \frac{3}{5}a.} \tag{5.14}$$

Furthermore, since $\vec{r}_0$ in (5.12) can be any point on the axis of the screw, we choose the point where the axis of the screw meets the horizontal plane, i.e., $z_0 = 0$. From (5.13), using (5.11) and (5.14):

$$\frac{ds}{3a}\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 2 & 1 \\ x_0 & y_0 & 0 \end{bmatrix} = \frac{4ds}{15}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \tag{5.15}$$

where $\hat{x}, \hat{y}, \hat{z}$ are the unit coordinate vectors. Solving for x_0, y_0, we obtain:

$$\boxed{x_0 = -\frac{4}{5}a, \qquad y_0 = 0, \qquad z_0 = 0.} \tag{5.16}$$

In summary, the tripod will begin to move about a right hand screw (5.12), whose pitch is given by (5.14), and whose axis meets the plane of the feet at a point $\frac{4}{5}a$ due west of the point O, as shown by (5.16). The axis, given by (5.11), is inclined $\tan^{-1} 2$ in the plane of the meridian (yz plane,) measured from the zenith (positive z axis) towards the north (positive y axis,) which completes the proof.

Question 6. If the centres of the bodies of the solar system are projected perpendicularly on a fixed plane, and the forces which act on them are resolved in that plane, show that the second differential coefficient with respect to the time of the moment of inertia of the system about an axis through its centre of mass perpendicular to that plane, together with twice the sum of the products of the projection of the distance between each pair of bodies into the resolved part of the attraction between them, is equal to four times the kinetic energy due to that part of the motion of the system relative to its centre of mass which is parallel to the plane of reference.

Solution by Raúl Simón and David Forfar

We assume all the planets in the solar system lie in a plane and the origin is the centre of mass of the planets. We take the perpendicular through that origin.
A body travelling at speed v relative to the origin has kinetic energy *(K.E.)* $\frac{1}{2}mv^2 = \frac{1}{2}m\frac{d\tilde{r}}{dt}.\frac{d\tilde{r}}{dt}$ where $\tilde{r}$ is the vector in the plane from the origin to the planet and we take the dot product of its velocity $\frac{d\tilde{r}}{dt}$. The moment of inertia of the body about the perpendicular at the origin is $I = m\tilde{r}.\tilde{r}$. Differentiating this once we have $\frac{dI}{dt} = 2m\tilde{r}.\frac{d\tilde{r}}{dt}$ and differentiating twice we have $\frac{d^2I}{dt^2} = 2m\frac{d\tilde{r}}{dt}.\frac{d\tilde{r}}{dt} + 2m\tilde{r}.\frac{d^2\tilde{r}}{dt^2}$. Using Newton's Law that vectorial force equals mass times vectorial acceleration, we have $\frac{d^2I}{dt^2} = 4.(K.E.) + 2\tilde{r}.\boldsymbol{F}$ where $\boldsymbol{F}$ is the vectorial force (i.e. the attraction from the other planets) acting on the planet. We therefore have, for planet i,

$$\frac{d^2I_i}{dt^2} - 2r_i.\boldsymbol{F}_i = 4.(K.E.)_i \tag{6.1}$$

We note that, in respect of any pair of planets, the forces are equal and opposite and thus if the force of planet j on planet i and is F_{ij} (this force points away from planet i) then the force of planet i on planet j is $\boldsymbol{F}_{ji} = -\boldsymbol{F}_{ij}$ and for convenience we define $\boldsymbol{F}_{ii} = 0$. Thus $\boldsymbol{F}_i = \sum_{j=1}^{j=n}\boldsymbol{F}_{ij}$. If we sum for all the n-planets (defining $I = \sum_{i=1}^{i=n} I_i$) as the total moment of inertia and *K.E.* as the total kinetic energy of the system (defining $K.E. = \sum_{i=1}^{i=n}(K.E.)_i$) we have:-

$$\frac{d^2I}{dt^2} - 2\sum_{i=1}^{i=n}\tilde{r}_i.\boldsymbol{F}_i = \frac{d^2I}{dt^2} - 2\sum_{i=1}^{i=n}\tilde{r}_i.(\sum_{j=1}^{j=n}\boldsymbol{F}_{ij}) = 4(K.E.) \tag{6.2}$$

Arranging the planets in pairs, we have :-

$$\frac{d^2 I}{dt^2} - 2\sum_{i=1}^{i=n}\sum_{j>i}^{j=n}(\tilde{r}_i - \tilde{r}_j).\boldsymbol{F}_{ij} = 4(K.E.) \qquad (6.3)$$

Remembering that $\boldsymbol{F}_{ij}$ points away from planet i towards planet j, whereas the vector $(\tilde{r}_i - \tilde{r}_j)$ points towards planet i from planet j, we have that the force $\boldsymbol{F}_{ij} = -G\frac{m_i * m_j}{|\tilde{r}_i - \tilde{r}_j|^3}(\tilde{r}_i - \tilde{r}_j)$ so the second term is $+2G\sum_{i=1}^{i=n}\sum_{j>i}^{j=n}|\tilde{r}_i - \tilde{r}_j|\frac{m_i * m_j}{|\tilde{r}_i - \tilde{r}_j|^2}$ which is the product of the distance between the planets and magnitude of their mutual attraction.

Question 7. A thin, uniformly elastic rod, *OABC*, originally straight, is constrained to pass through the points *A*, *B*, *C* in a straight line. Show that the deflection of the part *OA*, produced by forces acting on that part only, will be the same as if the rod had been constrained to pass through two points *A* and *X* only, where

$$AX = AB\frac{3AB + 4BC}{4AC}.$$

If the rod is constrained to pass through an infinite number of points, at intervals each equal to *AB*, show that the constraint, as regards the part *OA*, will be the same as if the rod had been constrained to pass through *A* and *Y*, where

$$AY = \frac{1}{2}\sqrt{3}AB.$$

Solution by Jovan Jevtic

Bending impedance. Consider a part of the rod whose end points are two adjacent points of constraint, such as A and B. The bending of the part AB is entirely determined by the bending moments at its end points, M_A and M_B. We'll refer to the points A and B as input and output, respectively. Let θ_A and θ_B be the angles which the bent rod makes with the line AB at the input and output, respectively. For small bending angles, a linear relationship exists between the input and output quantities:

$$\begin{bmatrix} M_A \\ \theta_A \end{bmatrix} = \mathbf{H}_{AB} \cdot \begin{bmatrix} M_B \\ \theta_B \end{bmatrix}, \quad \text{where: } \mathbf{H}_{AB} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \qquad (7.1)$$

where the matrix $\mathbf{H}_{AB}$ depends only on the distance AB and the elastic properties of the rod between the points A and B. Due to the similarity to the ray transfer matrix analysis in optics, we refer to the matrix $\mathbf{H}_{AB}$ as the bending transfer matrix between A and B.

As a special case, when there are no additional points of constraint beyond the point B, we have $M_B = 0$, and (7.1) gives:

$$Z_A \triangleq \left.\frac{M_A}{\theta_A}\right|_{M_B=0} = \frac{H_{12}}{H_{22}}, \qquad (7.2)$$

which may be interpreted as an input bending impedance of the part AB. Furthermore, if there are additional points of constraint beyond the point B, the effect of the rod beyond the point B may similarly be characterized by an input bending impedance $Z_B = M_B / \theta_B$ and used in (7.1) to obtain:

$$Z_A = \left.\frac{M_A}{\theta_A}\right|_{M_B = Z_B \theta_B} = \frac{H_{11} Z_B + H_{12}}{H_{21} Z_B + H_{22}}. \tag{7.3}$$

The last equation can be interpreted as the input bending impedance of the part AB loaded with the bending impedance Z_B at its output.

Bending transfer matrix. To derive an expression for the bending transfer matrix (7.1) for the part AB, we setup a Cartesian coordinate system x, y, z with the origin at the point A and the x axis pointing from A to B. We shall only consider the bending in the xy plane, since the bending analysis in the xz plane is identical. If $y(x)$ is the deflection of the rod, we recall from the theory of elasticity:

$$\frac{d^2 y}{dx^2} = -\frac{M(x)}{EI}, \tag{7.4}$$

where $M(x)$ is the bending moment, E is Young's modulus of elasticity for the material of the rod, and I is the moment of inertia of area of the rod's cross section. The bending moment varies linearly between the points A and B, $M(x) = M_A - N_A x$, where N_A is the reaction force of the constraining point A. Consequently:

$$M(x) = M_A + \frac{x}{AB}(M_B - M_A), \qquad 0 \le x \le AB. \tag{7.5}$$

The boundary conditions are:

$$y(0) = 0, \quad \left.\frac{dy}{dx}\right|_{x=0} = \theta_A, \quad y(AB) = 0, \quad \left.\frac{dy}{dx}\right|_{x=AB} = \theta_B \;. \tag{7.6}$$

A double integration of (7.4) from $x = 0$ to AB, with (7.1), (7.5), and (7.6) in mind, gives:

$$\mathbf{H}_{AB} = \begin{bmatrix} -2 & -\dfrac{6EI}{AB} \\ -\dfrac{AB}{2EI} & -2 \end{bmatrix}. \tag{7.7}$$

Distance AX. The input bending impedance of the parts AX and BC follow from (7.2) and (7.7), $Z_{AX} = 3EI / AX$ and $Z_{BC} = 3EI / BC$. The input bending impedance of the part ABC is that of part AB loaded with Z_{BC} and follows from (7.3) and (7.7):

$$Z_{ABC} = \frac{-2Z_{BC} - \dfrac{6EI}{AB}}{-\dfrac{AB}{2EI} Z_{BC} - 2} = \frac{-2\dfrac{3EI}{BC} - \dfrac{6EI}{AB}}{-\dfrac{AB}{2EI}\dfrac{3EI}{BC} - 2} = \frac{12EI(AB+BC)}{AB(3AB+4BC)}. \tag{7.8}$$

Part OA will deflect the same if $Z_{AX} = Z_{ABC}$ which finally gives:

$$\boxed{AX = AB\frac{3AB+4BC}{AB+BC}.} \tag{7.9}$$

Distance AY. If Z_N is the input bending impedance of the rod constrained to pass thru N equidistant points at intervals AB, then Z_N equals the input impedance of the part AB loaded at the output by the bending impedance Z_{N-1}. From (7.3) and (7.7):

$$Z_N = \frac{-2Z_{N-1} - \dfrac{6EI}{AB}}{-\dfrac{AB}{2EI} Z_{N-1} - 2} \quad \rightarrow \quad Z_N Z_{N-1} = \frac{4EI}{AB}(Z_{N-1} - Z_N) + \frac{12E^2I^2}{AB^2}. \tag{7.10}$$

Letting $N \to \infty$, we have $Z_{N-1} \to Z_N \to Z_\infty$, so (7.10) gives: $Z_\infty = 2\sqrt{3}\,EI/AB$. On the other hand, the input bending impedance of the part AY follows from (7.2) and (7.7), $Z_{AY} = 3EI / AY$. Part OA will deflect the same if $Z_{AY} = Z_\infty$ which requires:

$$\boxed{AY = \frac{\sqrt{3}}{2} AB,} \tag{7.11}$$

thus completing the proof.

Question 8. The configuration of four particles, whose masses are *P*, *Q*, *R*, *S*, is determined by their distances *QR = a*, *RP = b*, *PQ=c*, *PS=l*, *QS=m*, *RS=n*, and the potential energy of the system is

$$V = \tfrac{1}{2}\left[A(a-a_0)^2 + B(b-b_0)^2 + C(c-c_0)^2 + L(l-l_0)^2 + M(m-m_0)^2 + N(n-n_0)^2\right]:$$

show that the small oscillations of the system are determined by six equations of the form

$$QR\ddot{a} + (Q+R)A(a-a_0) + QB(b-b_0)\cos ab + RC(c-c_0)\cos ac$$
$$+ RM(m-m_0)\cos am + QN(n-n_0)\cos an = 0$$

where cos *ab* denotes the cosine of the angle *QRP* between *a* and *b*.

Show also that if the particles are all equal, and the law of force such that any two of them would be in equilibrium at a distance *a* and would make small oscillations of period *T*, then for three such particles the periods of the fundamental vibrations are $\sqrt{\tfrac{4}{3}}$ *T* and $\sqrt{\tfrac{2}{3}}$ *T*, and for four such particles $\sqrt{2}$ *T*, *T*, and $\tfrac{1}{2}\sqrt{2}$ *T*.

Solution by Jovan Jevtic

Let x_R, y_R, z_R be the rectangular coordinates of the particle of mass R and similarly for the other particles. The kinetic energy of the particle of mass R is then:

$$T_R = \frac{1}{2} R(\dot{x}_R^2 + \dot{y}_R^2 + \dot{z}_R^2), \tag{8.1}$$

and the Lagrangian, L, of the system is given by:

$$L = T_P + T_Q + T_R + T_S - V. \tag{8.2}$$

The equation of motion for the coordinate x_R is, from Lagrange's equations:-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_R}\right)-\frac{\partial L}{\partial x_R}=0 \to \frac{d}{dt}\left(\frac{\partial T_R}{\partial \dot{x}_R}\right)+\frac{\partial V}{\partial x_R}=0 \to R\ddot{x}_R+\frac{\partial V}{\partial a}\frac{\partial a}{\partial x_R}+\frac{\partial V}{\partial b}\frac{\partial b}{\partial x_R}+\frac{\partial V}{\partial n}\frac{\partial n}{\partial x_R}=0 \quad (8.3)$$

because the distances c,l,m are not affected by the displacement of the particle R alone. Since $a=\sqrt{(x_R-x_Q)^2+(y_R-y_Q)^2+(z_R-z_Q)^2}$, $\partial a/\partial x_R=(x_R-x_Q)/a$, (8.3) becomes:

$$R\ddot{x}_R+A(a-a_0)\frac{x_R-x_Q}{a}+B(b-b_0)\frac{x_R-x_P}{b}+N(n-n_0)\frac{x_R-x_S}{n}=0\,. \quad (8.4)$$

In a similar manner, we obtain the equation of motion for the coordinate x_Q :

$$Q\ddot{x}_Q+A(a-a_0)\frac{x_Q-x_R}{a}+M(m-m_0)\frac{x_Q-x_S}{m}+C(c-c_0)\frac{x_Q-x_P}{c}=0. \quad (8.5)$$

At this point of the analysis, we note that the orientation of the x axis in (8.4) and (8.5) is arbitrary. At a given instant of time, let us choose the x axis parallel to the line which passes thru the instantaneous positions of the particles of mass Q and R . This gives us:-

$$\frac{x_R-x_Q}{a}=1,\ \frac{x_R-x_P}{b}=\cos ab,\ \frac{x_R-x_S}{n}=\cos an,\ \frac{x_S-x_Q}{m}=\cos am,\ \frac{x_P-x_Q}{c}=\cos ac\,. \quad (8.6)$$

We now look at the difference of (8.4) and (8.5), with the identities (8.6) in mind:

$$\begin{aligned}\ddot{x}_R-\ddot{x}_Q+\left(\frac{1}{R}+\frac{1}{Q}\right)A(a-a_0)+\frac{B(b-b_0)}{R}\cos ab+\frac{N(n-n_0)}{R}\cos an+&\\ +\frac{M(m-m_0)}{Q}\cos am+\frac{C(c-c_0)}{Q}\cos ac=0.&\end{aligned} \quad (8.7)$$

Although $x_R-x_Q=a$ at the instant when the x axis is parallel to QR , we cannot, in general, assume that $\ddot{x}_R-\ddot{x}_Q=\ddot{a}$, due to the relative rotation between the line QR and the fixed x axis. In particular, if $\vec{a}=\overrightarrow{QR}$, it can be shown that:

$$\ddot{x}_R-\ddot{x}_Q=\ddot{a}-\frac{\dot{\vec{a}}^2-\dot{a}^2}{a}\,. \quad (8.8)$$

In the case of small amplitude oscillations, however, the terms in (8.8) which are proportional to the square of the amplitude may be ignored and we obtain from (8.7):

$$\begin{aligned}QR\ddot{a}+(Q+R)A(a-a_0)+QB(b-b_0)\cos ab+QN(n-n_0)\cos an+&\\ +RM(m-m_0)\cos am+RC(c-c_0)\cos ac=0\,,&\end{aligned} \quad (8.9)$$

exactly as specified in the problem statement. This result was obtained by starting with the equations of motion (8.4) and (8.5) for the pair of particles which define the distance a . Additional 5 equations may be derived in a similar manner by considering the pairs of particles which define the distances $b,c,l,m,$ and n .

Two particles. Setting $B=N=M=C=0$ and $Q=R$ in (8.9), gives $Q\ddot{a}+2A(a-a_0)=0$, which allows an oscillatory solution $a=a_0+a_m\cos\omega t$, provided that the angular frequency ω satisfies the eigenvalue equation $(-Q\omega^2+2A)a_m=0$. From $T=2\pi/\omega$:

$$\boxed{T = \pi\sqrt{\frac{2Q}{A}}}. \tag{8.10}$$

Three particles. Symmetry implies that the equilibrium positions are the vertices of an equilateral triangle. Setting $a_0 = b_0 = c_0$, $\cos ab = \cos ac = \cos(\pi/3) = 1/2$, $A = B = C,$ $N = M = 0,$ and $Q = R$ in (8.9), gives $Q\ddot{a} + 2A(a - a_0) + A(b - a_0)/2 + A(c - a_0)/2 = 0$. Assuming $a = a_0 + a_m \cos\omega' t,$ $b = a_0 + b_m \cos\omega' t,$ and $c = a_0 + a_m \cos\omega' t,$ we obtain an eigenvalue equation $(-Q\omega'^2 + 2A)a_m + Ab_m/2 + Ac_m/2 = 0$. Repeating for the remaining 2 pairs of particles:

$$\begin{bmatrix} 4-\lambda' & 1 & 1 \\ 1 & 4-\lambda' & 1 \\ 1 & 1 & 4-\lambda' \end{bmatrix} \cdot \begin{bmatrix} a_m \\ b_m \\ c_m \end{bmatrix} = 0, \quad \text{where: } \lambda' = \frac{2Q}{A}\omega'^2 = \left(\frac{2T}{T'}\right)^2 \tag{8.11}$$

and we've used $\omega' = 2\pi/T'$ and (8.10). Equating the determinant of the matrix (8.11) with zero, $(\lambda' - 3)^2(6 - \lambda') = 0$, leads to the desired periods of oscillation:

$$\lambda_1' = \lambda_2' = 3 \;\rightarrow\; \boxed{T_1' = T_2' = \frac{2}{\sqrt{3}}T,} \qquad \lambda_3' = 6 \;\rightarrow\; \boxed{T_3' = \sqrt{\frac{2}{3}}T.} \tag{8.12}$$

Four particles. Symmetry implies that the equilibrium positions are the vertices of a tetrahedron. Setting $a_0 = b_0 = c_0 = m_0 = n_0$, $A = B = C = M = N$, $Q = R$, and $\cos ab = \cos ac = \cos am = \cos an = \cos(\pi/3) = 1/2$ in (8.9), and assuming that a_m, b_m, c_m, m_m, n_m are the amplitudes of the oscillation of angular frequency ω'', we obtain $-Qa_m\omega''^2 + 2Aa_m + Ab_m/2 + Ac_m/2 + Am_m/2 + An_m/2 = 0$. Repeating for the remaining 5 pairs of particles:-

$$\begin{bmatrix} 4-\lambda'' & 1 & 1 & 0 & 1 & 1 \\ 1 & 4-\lambda'' & 1 & 1 & 0 & 1 \\ 1 & 1 & 4-\lambda'' & 1 & 1 & 0 \\ 0 & 1 & 1 & 4-\lambda'' & 1 & 1 \\ 1 & 0 & 1 & 1 & 4-\lambda'' & 1 \\ 1 & 1 & 0 & 1 & 1 & 4-\lambda'' \end{bmatrix} \cdot \begin{bmatrix} a_m \\ b_m \\ c_m \\ l_m \\ m_m \\ n_m \end{bmatrix} = 0, \text{ where: } \lambda'' = \frac{2Q}{A}\omega''^2 = \left(\frac{2T}{T''}\right)^2 \tag{8.13}$$

and we've used $\omega'' = 2\pi/T''$ and (8.10). Equating the determinant of the matrix (8.13) with zero, $(\lambda'' - 2)^2(\lambda'' - 4)^3(\lambda'' - 8) = 0$, leads to the desired periods of oscillation:

$$\lambda_{1,2}'' = 2 \rightarrow \boxed{T_{1,2}'' = \sqrt{2}T,} \qquad \lambda_{3,4,5}'' = 4 \rightarrow \boxed{T_{3,4,5}'' = T,} \qquad \lambda_6'' = 8 \rightarrow \boxed{T_6'' = \frac{1}{\sqrt{2}}T,} \tag{8.14}$$

which completes the proof.

Question 9. Define the principal foci and the principal focal length of an optical instrument, and show that in any system of thin lenses having the same axis:–

(1) The reciprocal of the principal focal length is the sum of the reciprocals of the focal lengths of the lenses, together with the sum of all intervals and products of consecutive intervals into which the axis may be divided by the lenses, each product being divided by the product of the focal lengths of the lenses at the points of section, including the first and last.

(2) The distance of the first principal focus of the instrument from the first lens is equal to the principal focal length of the system multiplied by 1 + the sum of all intervals and products of consecutive intervals beginning with the first lens, each divided by the product of the focal lengths of the lenses at the points of section excluding the first lens.

In what direction is this distance to be measured?

Solution by Winifred Sillitto, Douglas Essex and Jovan Jevtic

Definitions

(a) The **first principal focal point** (of a rotationally invariant optical instrument) is the point on the central (optical) axis of the instrument such that all light rays (incident on the optical instrument) passing through the **first principal focal point** emerge from the optical instrument parallel to the axis, The **second principal focal point** (of a rotationally invariant optical instrument) is the point on the central (optical) axis of the instrument such that all light rays parallel to the axis (and incident on the optical instrument) pass through the **second principal focal point** on emerging from the optical instrument,.

(b) the **first principal plane** is the transverse plane where a ray of incident light coming from the **first principal focal point** would intersect the extended line of the same light ray, now emerging parallel to the axis. The **second principal plane** is the transverse plane where the line of a ray parallel to the axis (and incident on the instrument) would intersect the extended line of the same ray now emerging and going through the **second principal focal point**.

(c) The **first principal focal length** is the distance (along the central axis) between the **first principal focal point** and the **first principal plane.** The **second principal focal length** is the distance (along the central axis) between the **second principal plane** and the **second principal focal point**.

[Note:- There are **two principal planes** which are planes of unit magnification and can be considered as the boundaries of an equivalent ideal thin lens. We assume the refractive index is the same on both sides of the instrument and so the **first** and **second principal focal lengths** are equal**.** We call the **focal length** of the optical instrument, *F*].

Matrix representation of an optical instrument

An incident light ray (coming in to the optical instrument from the right hand side) which intersects the central axis of the optical instrument, may be described by a pair of co-ordinates, y and α, where:-

(i) *y* is the displacement in a plane transverse to the (optical) central axis between the point where the axis intersects the plane and the point where the ray intersects the plane,
(ii) α is the angle of the light ray, as it intersects that transverse plane, relative to the central axis measured in the plane which contains both the ray and the axis.

We arbitrarily treat as positive (1) displacements from the central axis and (2) clockwise rotations between a vector pointing in the direction of travel of the ray of light towards the optical instrument and a vector pointing along the central axis towards the optical instrument and (3) distances from the first lens in the direction of the incident ray.

We use the assumptions (paraxial) that α=sin α= tan α and cos α=1, and that the lens is so thin that it can be treated as a pair of contiguous planes; also that the refractive index is unity outside and between the lenses.

As a ray is transferred from one transverse plane to another transverse plane, with (y, α) for the ray at the first plane and (y', α') for the ray at the second plane, we have (by trigonometry) two special cases:-

(a) refraction - from a plane just in front of an ideal thin lens of focal length f to a plane just behind the lens (as seen from the source of the incident light):

A single lens (of negligible width) of focal length f :-

$$y' = y$$

$$\alpha' = -\frac{y}{f} + \alpha \qquad \text{or}$$

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix} \tag{9.1}$$

where the power, p, of the lens is defined as $1/f$

(b) translation -(in the direction of travel of the light) across a lens-free gap (or space) of axial length d :-

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix} \text{ or } \begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix} \tag{9.2}$$

In general the behaviour of a light ray between any two planes (transverse to the axis of an optical instrument) can be described by a matrix, where :-

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} y \\ \alpha \end{pmatrix} \tag{9.3}$$

<u>For transfer from an incident ray parallel to the axis to the corresponding emergent ray through the **second principal focal point** :-</u>

*α=0, thus $\alpha'=m_{21}y=-y/F=-Py$, where F is the **principal focal length** of the optical instrument and P is the power of the optical instrument and we have $P=-m_{21}$*

<u>For transfer from an incident ray through the **first principal focal point** to an emergent ray parallel to the optical axis.</u>

*α =y/D and α' = 0 where D (positive) is the distance from the **first lens** to the **first principal focal point**, then*

$$m_{21} + m_{22}\frac{y}{D} = 0$$

and so

$$D = -\frac{m_{22}}{m_{21}} = F.m_{22}$$

Thus to find expressions for P and D we need to determine m_{21} and m_{22}.

A coaxial system consisting of a lens of power p_1 , followed by a gap of length d_1, followed by a lens of power p_2 gives:-

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p_1 & 1 \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} 1-p_1d_1 & d_1 \\ -p_1-p_2+p_1p_2d_1 & 1-p_2d_1 \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix} \quad (9.4)$$

Therefore $P = \frac{1}{F} = -m_{21} = p_1 + p_2 - p_1p_2d_1$ and $D = F.m_{22} = F.(1-p_2d_1)$

For 3 lenses and 2 gaps we have:-

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p_3 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p_1 & 1 \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix} \quad (9.5)$$

The matrix is:-

$$\begin{pmatrix} 1-p_1d_1-p_1d_2-p_2d_2+p_1p_2d_1d_2 & d_1+d_2-p_2d_1d_2 \\ -p_1-p_2-p_3+p_1p_2d_1+p_1p_3d_1+p_1p_3d_2+p_2p_3d_2-p_1p_2p_3d_1d_2 & 1-p_2d_1-p_3d_1-p_3d_2+p_2p_3d_1d_2 \end{pmatrix} \quad (9.6)$$

So $P = \frac{1}{F} = -m_{21} = p_1 + p_2 + p_3 - (p_2 + p_3)p_1d_1 - (p_1 + p_2)p_3d_2 + p_1p_2p_3d_1d_2$

$D = F.m_{22} = F.\{1-(p_2 + p_3)d_1 - p_3d_2 + p_2p_3d_1d_2)$

For 4 lenses and 3 gaps we have:-

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p_4 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p_3 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -p_1 & 1 \end{pmatrix}\begin{pmatrix} y \\ \alpha \end{pmatrix} \quad (9.7)$$

Multiplying these matrices out and thus determining m_{21} and m_{22}:-

$$
\begin{aligned}
P &= \frac{1}{F} = -m_{21} = p_1 + p_2 + p_3 + p_4 \\
&-(p_2+p_3+p_4)p_1 d_1 - (p_1 p_3 + p_2 p_3 + p_2 p_4 + p_1 p_4)d_2 - (p_1+p_2+p_3)p_4 d_3 \\
&+p_1 p_2 p_3 d_1 d_2 + (p_2+p_3)p_1 p_4 d_1 d_3 + (p_1 p_2 + p_1 p_3 + p_2 p_3)p_4 d_2 d_3 \\
&-p_1 p_2 p_3 p_4 d_1 d_2 d_3
\end{aligned}
\tag{9.8}
$$

$$
\begin{aligned}
D = F.m_{22} = F.\{1 &- (p_2+p_3+p_4)d_1 - (p_3+p_4)d_2 - p_4 d_3 \\
&+p_2 p_3 d_1 d_2 + (p_2+p_3)p_4 d_1 d_3 + (p_2+p_3)p_4 d_2 d_3 \\
&-p_2 p_3 p_4 d_1 d_2 d_3)\}
\end{aligned}
\tag{9.9}
$$

and we can see how an additional lens adds terms to the effective power of the lens assembly constituting the optical instrument.

Examining the expressions for *P* and *D* we see that they can be interpreted as stated by Clerk Maxwell allowing for the fact that we have taken a converging lens to have positive focal length but the opposite sign convention is also valid (Maxwell, by his own admission, was not too hot on signs!).

Considering the definitions above, we see that it is the distances between the lenses which count..

If all the distances between the lenses are zero (i.e. the lenses are on top of one another) we have, as expected,

$$
\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} \ldots\ldots \tag{9.10}
$$

<u>For more information, see:-</u>

Sampson R. A. (1913), *A New Treatment of Optical Aberrations*, Philosophical Transactions of the Royal Society, **121**, pp. 149-185.

Gerrard A. and Burch J. M. (1994), *Introduction to Matrix Methods in Optics*, Dover Publications.

Longhurst R. S. (1957), *Geometrical and Physical Optics*, Longmans.

Question 10. The motion of an incompressible homogeneous fluid in a spherical vessel at a given instant is such that each spherical stratum rotates like a rigid shell, the rectangular components of its angular velocity being ω_1, ω_2, ω_3, these quantities varying from stratum to stratum: show that if each particle is attracted towards the centre with a force whose intensity per unit of mass is

$$
(\varpi_1 x + \varpi_2 y + \varpi_3 z)\left(x\frac{d\varpi_1}{dr} + y\frac{d\varpi_2}{dr} + z\frac{d\varpi_3}{dr} \right) + \frac{dV}{dr},
$$

where *V* is any function of the co-ordinates, the motion of the fluid will be steady, and determine the pressure at any point.

Solution by Jovan Jevtic

The motion of an incompressible homogeneous fluid is governed by the equations:

$$div\,\vec{\mathbf{v}} = 0 \text{ (conservation of mass)}$$
$$\frac{\partial\vec{\mathbf{v}}}{\partial t} + \left(\vec{\boldsymbol{v}}\cdot grad\right)\vec{\mathbf{v}} = \vec{\boldsymbol{f}} - \frac{1}{\rho}grad\ p \quad \text{(Euler's equation)} \tag{10.1}$$

where $\vec{\boldsymbol{v}}$ is the velocity, $\vec{\boldsymbol{f}}$ is the force per unit mass, p is the pressure, and ρ is the density of the fluid. Let $\vec{\boldsymbol{r}}$ be the radius vector of a fluid particle and let $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}}(r)$ be the angular velocity vector of a spherical stratum of radius r. Knowing that the fluid particles at the given instant move along circular trajectories, $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}$, we first confirm that the equation of continuity in (10.1) is satisfied:

$$div\,\vec{\mathbf{v}} = \frac{\partial}{\partial x}(\omega_2 z - \omega_3 y) + \frac{\partial}{\partial y}(\omega_3 x - \omega_1 z) + \frac{\partial}{\partial z}(\omega_1 y - \omega_2 x) \equiv 0\,, \tag{10.2}$$

and second, we have that:

$$\left(\vec{\boldsymbol{v}}\cdot grad\right)\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}}\times(\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}) = (\vec{\boldsymbol{\omega}}\cdot\vec{r})\vec{\omega} - \omega^2\vec{\boldsymbol{r}}\,. \tag{10.3}$$

Furthermore, the force given in the problem statement can be written in the vector form:

$$\boxed{\vec{\boldsymbol{f}} = -(\vec{\boldsymbol{\omega}}\cdot\vec{\boldsymbol{r}})\left(\vec{\boldsymbol{r}}\cdot\frac{d\vec{\boldsymbol{\omega}}}{dr}\right)\hat{\boldsymbol{r}} - grad\,V,} \tag{10.4}$$

where $\hat{\boldsymbol{r}} = \vec{\boldsymbol{r}}/r$ stands for the unit vector. When (10.3) and (10.4) are used in (10.1), we obtain:

$$\frac{\partial\vec{\mathbf{v}}}{\partial t} = \omega^2\vec{\boldsymbol{r}} - (\vec{\boldsymbol{\omega}}\cdot\vec{r})\left[\vec{\boldsymbol{\omega}} + \left(\vec{\boldsymbol{r}}\cdot\frac{d\vec{\boldsymbol{\omega}}}{dr}\right)\hat{\boldsymbol{r}}\right] - grad\left(V + \frac{p}{\rho}\right). \tag{10.5}$$

If $\phi(r) = \int_{u=0}^{u=r}\omega^2(u)u\,du - \frac{1}{2}(\vec{\omega}.\vec{r})^2$ then we obtain:

$$\boxed{\frac{\partial\vec{v}(r,t)}{\partial t} = grad\left(\phi(r) + V(r) + \frac{p(r,t)}{\rho} + c\right)} \tag{10.6}$$

Where c is a constant.

Is there a pressure distribution $p(r,t)$ that leads to a steady flow of fluid,

i.e., $\partial\vec{v}(r,t)/\partial t = 0$ in (10.6)?

If

$$p(r,t) = \rho(-V(r) - \phi(r) - c) \tag{10.7}$$

then $\dfrac{\partial\vec{v}(r,t)}{\partial t} = 0$ everywhere, which completes the solution.

Comment by Professor H.K. Moffatt:-

This problem is a little artificial, as it involves a hypothetical force distribution which is just such as to maintain the assumed (non-viscous) fluid flow.

This type of flow does actually occur in a viscous fluid contained between two concentric spheres, if the spheres rotate about different axes through their common centre with different angular velocities. If the Reynolds number is *small, and inertia is neglected (the opposite of the situation considered by Maxwell), then the fluid does indeed move with purely tangential velocity on spherical shells. Inertia perturbs this flow however and drives meridional circulation, which is a considerable complication!*

See reference:- Bajer K. & Moffatt H.K. (1992), Chaos associated with fluid inertia. In: Topological Aspects of the Dynamics of Fluids and Plasmas. Editors, Moffatt H.K., Zaslavsky G.M., Tabor M. and Comte P. Kluwer Academic Publishers, pp. 517-534.

Question 11. Show that if *P* be a variable point on a sphere, and *A*, *B*, fixed points,

$$3\cos PA\cos PB - \cos AB$$

is a spherical harmonic of the second order, and that if *A′*, *B′*, are the poles of another harmonic, the condition of the two harmonics being conjugate to each other is

$$3\cos AA'\cos BB' + 3\cos AB'\cos A'B - 2\cos AB\cos A'B' = 0$$

If the intersections of five equidistant meridians with two parallels of latitude are the poles of five spherical harmonics of the second order, show that if they are all conjugate to each other, the polar distances of the two circles must be $\tan^{-1}\left(2+\sqrt{6}\pm\sqrt{14+6\sqrt{6}}\right)$.

Solution by Jovan Jevtic

Spherical Harmonics. We use a notation closely paralleling Mawell's exposition on spherical harmonics in the 3rd edition of the *Treatise* (Chapter IX, Articles 128 thru 133.) Let O be the center of the sphere, $r = OP$, $\mu_a = \cos PA$, $\lambda_{ab} = \cos AB$, and let $\partial/\partial h_a$ be the derivative in the direction OA. Then:

$$\partial r/\partial h_a = \mu_a,\ \partial\mu_b/\partial h_a = \left(\lambda_{ab} - \mu_a\mu_b\right)/r,\ \partial\mu_a/\partial h_a = \left(1-\mu_a^2\right)/r,\ \partial\lambda_{ab}/\partial h_a = 0\,, \quad (11.1)$$

(*ibid.*, equations 5 and 7.) It suffices to prove that:

$$\nabla^2\Psi = \partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 + \partial^2\Psi/\partial z^2 = 0 \quad (11.2)$$

for:

$$\boxed{\Psi = \left(3\mu_a\mu_b - \lambda_{ab}\right)r^2.} \quad (11.3)$$

Consider first the derivative with respect to x, $\partial/\partial x = \partial/\partial h_x$:

$$\begin{aligned}\frac{\partial\psi}{\partial h_x} &= \left(3\frac{\lambda_{ax}-\mu_a\mu_x}{r}\mu_b + 3\mu_a\frac{\lambda_{bx}-\mu_b\mu_x}{r}\right)r^2 + 2r\mu_x\left(3\mu_a\mu_b - \lambda_{ab}\right) = \\ &= r(3\mu_a\lambda_{bx} + 3\mu_b\lambda_{ax} - 2\mu_x\lambda_{ab}),\end{aligned} \quad (11.4)$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial h_x^2} &= \mu_x(3\mu_a\lambda_{bx} + 3\mu_b\lambda_{ax} - 2\mu_x\lambda_{ab}) + \\ &\quad + r\left(3\frac{\lambda_{ax}-\mu_a\mu_x}{r}\lambda_{bx} + 3\frac{\lambda_{bx}-\mu_b\mu_x}{r}\lambda_{ax} - 2\frac{1-\mu_x^2}{r}\lambda_{ab}\right) = 6\lambda_{ax}\lambda_{bx} - 2\lambda_{ab}.\end{aligned} \quad (11.5)$$

When (11.5) is used in (11.2), we obtain:

$$\boxed{\nabla^2\Psi = 6(\lambda_{ax}\lambda_{bx} + \lambda_{ay}\lambda_{by} + \lambda_{az}\lambda_{bz}) - 6\lambda_{ab} = 0}, \tag{11.6}$$

which shows that $\left(3\mu_a\mu_b - \lambda_{ab}\right)r^2$ satisfies Laplace's equation and that $3\mu_a\mu_b - \lambda_{ab}$ is indeed a spherical harmonic of the second order.

Conjugate Harmonics. Let $Y_{ab} = (3\mu_a\mu_b - \lambda_{ab})/2$ and $Y_{a'b'} = (3\mu_{a'}\mu_{b'} - \lambda_{a'b'})/2$ be the spherical harmonics corresponding to the poles A, B and A', B', respectively. If Y_{ab} and $Y_{a'b'}$ are conjugate to each other, we must have:

$$\iint_{r=a} Y_{ab}Y_{a'b'}ds = 0\,. \tag{11.7}$$

In a beautiful derivation, rich with physical insight, and to which the reader is referred (*ibid.*, equation 31,) Maxwell has shown that, in general:

$$\iint_{r=a} Y_{a_1\ldots a_n}Y_{b_1\ldots b_m}ds = \frac{4\pi}{n!(2n+1)}a^{n-m+2}\left.\frac{\partial^n(r^mY_{b_1\ldots b_m})}{\partial h_{a_1}\ldots\partial h_{a_n}}\right|_{r=0}, \tag{11.8}$$

where the poles of the spherical harmonics of orders n and m are $A_1,\ldots,A_n$ and $B_1,\ldots,B_m$. This equation reduces the integration over a sphere of radius a to a differentiation at the origin. In our case, $n = m = 2,\ A_1A_2 \to A'B',\ B_1B_2 \to AB$, so (11.7) and (11.8) reduce to:

$$\iint_{r=a} Y_{ab}Y_{a'b'}ds = 0 \quad\to\quad \left.\frac{\partial^2(r^2Y_{ab})}{\partial h_{a'}\partial h_{b'}}\right|_{r=0} = 0 \quad\to\quad \left.\frac{\partial^2\Psi}{\partial h_{a'}\partial h_{b'}}\right|_{r=0} = 0 \tag{11.9}$$

where we've already defined ψ in (11.3). The derivative $\partial/\partial h_{a'}$ follows directly from (11.4) by substituting a' for x:

$$\frac{\partial\psi}{\partial h_{a'}} = r(3\mu_a\lambda_{ba'} + 3\mu_b\lambda_{aa'} - 2\mu_{a'}\lambda_{ab}). \tag{11.10}$$

The additional derivative $\partial/\partial h_{b'}$ can be evaluated with the aid of (11.1):

$$\begin{aligned}\frac{\partial^2\Psi}{\partial h_{b'}\partial h_{a'}} &= \mu_{b'}(3\mu_a\lambda_{ba'} + 3\mu_b\lambda_{aa'} - 2\mu_{a'}\lambda_{ab}) + \\ &\quad + r\left(3\frac{\lambda_{ab'} - \mu_a\mu_{b'}}{r}\lambda_{ba'} + 3\frac{\lambda_{bb'} - \mu_b\mu_{b'}}{r}\lambda_{aa'} - 2\frac{\lambda_{a'b'} - \mu_{a'}\mu_{b'}}{r}\lambda_{ab}\right) = \\ &= 3\lambda_{aa'}\lambda_{bb'} + 3\lambda_{ab'}\lambda_{a'b} - 2\lambda_{a'b'}\lambda_{ab}\,.\end{aligned} \tag{11.11}$$

Consequently, the condition (11.9) reduces to:

$$\boxed{3\lambda_{aa'}\lambda_{bb'} + 3\lambda_{ab'}\lambda_{a'b} - 2\lambda_{ab}\lambda_{a'b'} = 0,} \tag{11.12}$$

exactly as specified in the problem.

Consider 10 poles whose polar distances θ and azimuths ϕ are given by:

$$A_n = (\theta_A, n\phi),\ \ B_n = (\theta_B, n\phi), \qquad n = 0,1,2,3,4\,, \tag{11.13}$$

where $\phi = 2\pi/5$. These poles define 5 spherical harmonics of the second order by the equations $Y_{a_n b_n} = 3\mu_{a_n}\mu_{b_n} - \lambda_{a_n b_n}$, $n = 0,1,2,3,4$, as shown in the first part of the problem. If $Y_{a_n b_n}$ is to be conjugate to $Y_{a_m b_m}$, the condition (11.12) must be satisfied:

$$3\lambda_{a_n a_m}\lambda_{b_n b_m} + 3\lambda_{a_n b_m}\lambda_{a_m b_n} - 2\lambda_{a_n b_n}\lambda_{a_m b_m} = 0, \qquad n \neq m \tag{11.14}$$

From (11.13), we obtain:

$$\begin{aligned}
\lambda_{a_n a_m} &= \sin^2\theta_A \cos(n-m)\phi + \cos^2\theta_A, \\
\lambda_{a_n b_m} &= \sin\theta_A \sin\theta_B \cos(n-m)\phi + \cos\theta_A \cos\theta_B, \\
\lambda_{b_n b_m} &= \sin^2\theta_B \cos(n-m)\phi + \cos^2\theta_B,
\end{aligned} \tag{11.15}$$

so that (11.14) becomes:

$$\begin{aligned}
&3[\sin^2\theta_A \cos(n-m)\phi + \cos^2\theta_A][\sin^2\theta_B \cos(n-m)\phi + \cos^2\theta_B] + \\
&+3[\sin\theta_A \sin\theta_B \cos(n-m)\phi + \cos\theta_A \cos\theta_B]^2 - 2[\sin\theta_A \sin\theta_B + \cos\theta_A \cos\theta_B]^2 = 0.
\end{aligned} \tag{11.16}$$

To simplify, divide by $\sin^2\theta_A \sin^2\theta_B$ and define:

$$x_A = 1/\tan\theta_A, \quad x_B = 1/\tan\theta_B \tag{11.17}$$

Then (11.16) reduces to:

$$6\cos^2(n-m)\phi + 3(x_A + x_B)^2 \cos(n-m)\phi + 4x_A^2 x_B^2 - 4x_A x_B - 2 = 0. \tag{11.18}$$

which can be solved for $\cos(n-m)\phi$:

$$\cos(n-m)\phi = \left[-3(x_A + x_B)^2 \pm \sqrt{9(x_A + x_B)^4 - 48(2x_A^2 x_B^2 - 2x_A x_B - 1)}\right]/12 \tag{11.19}$$

Given the polar distances of the poles A and B, x_A and x_B are fixed by (11.17), and we see that the orthogonality condition (11.19) allows only two possible values of $\cos(n-m)\phi$ for any combination of $n \neq m$. This is indeed the case for $\phi = 2\pi/5$, because:

$$\cos(n-m)\phi = \begin{cases} (-1+\sqrt{5})/4, & |n-m| = 1 \text{ or } 4 \\ (-1-\sqrt{5})/4, & |n-m| = 2 \text{ or } 3 \end{cases} \tag{11.20}$$

Comparing (11.19) to (11.20) gives:

$$(x_A + x_B)^2 = 1 \quad \text{and} \quad \sqrt{9(x_A + x_B)^4 - 48(2x_A^2 x_B^2 - 2x_A x_B - 1)} = 3\sqrt{5} \tag{11.21}$$

which simplifies to:

$$(x_A + x_B)^2 = 1 \quad \text{and} \quad 8x_A^2 x_B^2 - 8x_A x_B - 1 = 0, \tag{11.22}$$

and finally:

$$x_A + x_B = [\pm]1 \quad \text{and} \quad x_A x_B = (2\{\pm\}\sqrt{6})/4, \tag{11.23}$$

where we use the square and curly brackets to keep track of the alternative signs. Solving:

$$x_{A,B} = \left([\pm]1 \pm \sqrt{\sqrt{6} - 1}\right)/2, \tag{11.24}$$

where $\{\pm\}$ reduces to $-$ because $+$ gives an imaginary solution. Finally, we use (11.17):

$$\tan\theta_{A,B} = \frac{1}{x_{A,B}} = \frac{2}{[\pm]1 \pm \sqrt{\sqrt{6}-1}} \cdot \frac{[\pm]1 \mp \sqrt{\sqrt{6}-1}}{[\pm]1 \mp \sqrt{\sqrt{6}-1}} =$$
$$= 2 \cdot \frac{[\pm]1 \mp \sqrt{\sqrt{6}-1}}{2-\sqrt{6}} \cdot \frac{2+\sqrt{6}}{2+\sqrt{6}} = [\mp](2+\sqrt{6}) \pm \sqrt{\left(\sqrt{6}-1\right)\left(10+4\sqrt{6}\right)} = \tag{11.25}$$
$$= [\mp](2+\sqrt{6}) \pm \sqrt{14+6\sqrt{6}}$$

If the pair of polar distances θ_A, θ_B is a solution corresponding to the $+$ sign in $[\pm]$, then the pair $\pi-\theta_B, \pi-\theta_A$ is the solution corresponding to the $-$ sign, which is equivalent to a trivial interchange of the north and south poles on the sphere. We conclude, therefore, that the polar distances of the poles (11.13) are given by:

$$\boxed{\theta_{A,B} = \tan^{-1}\left(2+\sqrt{6} \pm \sqrt{14+6\sqrt{6}}\right),} \tag{11.26}$$

which completes the proof.

Question 12. A soap bubble is gradually charged with electricity; determine the pressure of the air within it, and show that when it becomes less than that of the air outside, the equilibrium of the bubble becomes unstable with respect to small deviations from the spherical form.

Solution by Dr. Jovan Jevtic

Surface tension. Let κ_1 and κ_2 be the principal curvatures of a small surface element of the bubble. The surface tension pulls on the rim of the element by a net force of the same intensity and direction as the force produced by an external pressure of magnitude:

$$p_s = \gamma(\kappa_1+\kappa_2) = \gamma\, div\hat{n}\,, \tag{12.1}$$

where γ is the coefficient of surface tension and $\hat{n}$ is the outside normal to the surface. For a spherical bubble of radius a, we have $\kappa_1 = \kappa_2 = 1/a$, so that (12.1) becomes:

$$p_s^{(0)} = \frac{2\gamma}{a}\,. \tag{12.2}$$

Electrostatic pressure. Since the bubble is an electrical conductor, the electric field is zero inside the bubble and normal to the outside surface of the bubble. The net electrostatic force acting on the element of surface is of the same intensity and direction as the force produced by an internal pressure of magnitude:

$$p_e = \frac{1}{2}\varepsilon_0\left(\frac{\partial V}{\partial n}\right)^2, \tag{12.3}$$

where V is the electrostatic potential and ε_0 is the permittivity of vacuum (air.) For a spherical bubble carrying a net electrical charge q:

$$V^{(0)}(r) = \frac{q}{4\pi\varepsilon_0 r}, \qquad r \ge a\,, \tag{12.4}$$

where r is the distance from the center, so that (12.3) becomes:

$$p_e^{(0)} = \frac{1}{32\pi^2\varepsilon_0}\frac{q^2}{a^4}\,. \tag{12.5}$$

Equilibrium. The pressure balance for a spherical bubble follows from (12.2) and (12.5):

$$\boxed{p = p_0 + p_s^{(0)} - p_e^{(0)} = p_0 + \frac{2\gamma}{a} - \frac{1}{32\pi^2\varepsilon_0}\frac{q^2}{a^4}}, \tag{12.6}$$

where p and p_0 is the pressure of the air within and outside the bubble, respectively.

Perturbation of the shape. Let $r^{(1)}$ be the deviation of the distance to the center:

$$r(\theta,\phi) = a + r^{(1)}(\theta,\phi)\,. \tag{12.7}$$

Following Maxwell (*Treatise* , 3rd ed., Art. 145a,) we expand the small deviations from the spherical form in a series of spherical harmonics:

$$\boxed{r^{(1)}(\theta,\phi) = a\sum_{n=2}^{\infty}\sum_{m=-n}^{n} r_{nm}Y_n^m(\theta,\phi),} \qquad |r_{nm}| \ll 1\,, \tag{12.8}$$

where r,θ,ϕ are the spherical coordinates of a point on the bubble. The $n=0$ and $n=1$ terms have been excluded because they only affect the mean radius and the position of the center of mass, respectively. We can find the unit vectors tangent to the surface:

$$\begin{aligned} \vec{u} &= \frac{\partial\vec{r}}{\partial\theta} = \frac{\partial r}{\partial\theta}\hat{r} + r\frac{\partial\hat{r}}{\partial\theta} = \frac{\partial r}{\partial\theta}\hat{r} + r\hat{\theta} \qquad \rightarrow \qquad \hat{u} = \hat{\theta} + \frac{1}{a}\frac{\partial r^{(1)}}{\partial\theta}\hat{r} + o(r_{nm}^2), \\ \vec{v} &= \frac{\partial\vec{r}}{\partial\phi} = \frac{\partial r}{\partial\phi}\hat{r} + r\frac{\partial\hat{r}}{\partial\phi} = \frac{\partial r}{\partial\phi}\hat{r} + r\sin\theta\hat{\phi} \qquad \rightarrow \qquad \hat{v} = \hat{\phi} + \frac{1}{a\sin\theta}\frac{\partial r^{(1)}}{\partial\phi}\hat{r} + o(r_{nm}^2), \end{aligned} \tag{12.9}$$

or, considering the expansion (12.8), to within $o(r_{nm}^2)$:

$$\begin{gathered} \hat{u} = \hat{\theta} + \hat{r}\sum_{n=2}^{\infty}\sum_{m=-n}^{n} r_{nm}\frac{\partial Y_n^m}{\partial\theta}, \quad \hat{v} = \hat{\phi} + \frac{\hat{r}}{\sin\theta}\sum_{n=2}^{\infty}\sum_{m=-n}^{n} r_{nm}\frac{\partial Y_n^m}{\partial\phi}, \\ \hat{n} = \hat{u}\times\hat{v} = \hat{r} - \sum_{n=2}^{\infty}\sum_{m=-n}^{n} r_{nm}\left(\hat{\theta}\frac{\partial Y_n^m}{\partial\theta} + \hat{\phi}\frac{1}{\sin\theta}\frac{\partial Y_n^m}{\partial\phi}\right). \end{gathered} \tag{12.10}$$

Perturbation of the potential. Let $V^{(1)}$ be the deviation of the electrostatic potential from the equilibrium value (12.4) :

$$V(r,\theta,\phi) = V^{(0)}(r) + V^{(1)}(r,\theta,\phi)\,. \tag{12.11}$$

Since it satisfies the Laplace's equation we can expand it in terms of solid harmonics:

$$V^{(1)}(r,\theta,\phi) = \frac{q}{4\pi\varepsilon_0}\sum_{n=2}^{\infty}\sum_{m=-n}^{n} v_{nm}\frac{a^n}{r^{n+1}}Y_n^m(\theta,\phi), \qquad |v_{nm}| \ll 1\,. \tag{12.12}$$

Additionally, the tangential components of the electric field must vanish on the surface of the conducting bubble. Keeping in mind the orders of magnitude given in (12.10):

$$\begin{aligned} \hat{u}\cdot gradV &= \frac{\partial V}{\partial r}u_r + \frac{1}{r}\frac{\partial V}{\partial\theta}u_\theta = \left.\frac{\partial V^{(0)}}{\partial r}\right|_{r=a} u_r + \left.\frac{1}{a}\frac{\partial V^{(1)}}{\partial\theta}\right|_{r=a} + o(r_{nm}^2) = \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\left[\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\left(v_{nm} - r_{nm}\right)\frac{\partial Y_n^m}{\partial\theta}\right] + o(r_{nm}^2) = 0\,, \end{aligned} \tag{12.13}$$

and similarly for $\hat{v}\cdot gradV = 0$. We conclude that:

$$v_{nm} = r_{nm}. \tag{12.14}$$

Perturbation of the electrostatic pressure. Keeping in mind $\hat{n}$ from (12.10):

$$\left(\frac{\partial V}{\partial n}\right)^2 = \left(\frac{\partial V}{\partial r}n_r + \frac{1}{r}\frac{\partial V}{\partial \theta}n_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}n_\phi\right)^2 = \left(\frac{\partial V^{(0)}}{\partial r} + \left.\frac{\partial V^{(1)}}{\partial r}\right|_{r=a} + o(r_{nm}^2)\right)^2 =$$
$$= \left(\left.\frac{\partial V^{(0)}}{\partial r}\right|_{r=a} + \left.\frac{\partial^2 V^{(0)}}{\partial r^2}\right|_{r=a} r^{(1)} + \left.\frac{\partial V^{(1)}}{\partial r}\right|_{r=a} + o(r_{nm}^2)\right)^2. \tag{12.15}$$

Consequently, the deviation of the electrostatic pressure (12.3) is given by:

$$p_e^{(1)} = \frac{1}{2}\varepsilon_0 2\left(\left.\frac{\partial V^{(0)}}{\partial r}\right|_{r=a}\right)\left(\left.\frac{\partial^2 V^{(0)}}{\partial r^2}\right|_{r=a} r^{(1)} + \left.\frac{\partial V^{(1)}}{\partial r}\right|_{r=a}\right) + o(r_{nm}^2), \tag{12.16}$$

which may be evaluated using (12.4), (12.8), and (12.12). We obtain:

$$p_e^{(1)} = \varepsilon_0\left(-\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\right)\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\left[2r_{nm} - (n+1)v_{nm}\right]Y_n^m\right). \tag{12.17}$$

Finally, due to (12.14):

$$\boxed{p_e^{(1)} = \frac{1}{16\pi^2\varepsilon_0}\frac{q^2}{a^4}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}(n-1)r_{nm}Y_n^m.} \tag{12.18}$$

Perturbation of the surface tension. Using $\hat{n}$ from (12.10) in (12.1), we obtain:

$$p_s = \frac{\gamma}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta) - \sum_{n=2}^{\infty}\sum_{m=-n}^{n} r_{nm}\left[\frac{\partial}{\partial\theta}\left(r\sin\theta\frac{\partial Y_n^m}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{r}{\sin\theta}\frac{\partial Y_n^m}{\partial\phi}\right)\right]\right\} =$$
$$= \frac{\gamma}{r}\left\{2 - \sum_{n=2}^{\infty}\sum_{m=-n}^{n} r_{nm}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_n^m}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_n^m}{\partial\phi^2}\right]\right\}. \tag{12.19}$$

The expression inside the square brackets simplifies to $-n(n+1)Y_n^m$. Furthermore:

$$\frac{1}{r} = \frac{1}{a} - \frac{1}{a^2}r^{(1)} + o(r_{nm}^2), \tag{12.20}$$

so that the deviation of the surface tension pressure obtains from (12.19) and (12.8):

$$p_s^{(1)} = -\frac{2\gamma}{a^2}r^{(1)} + \frac{\gamma}{a}\sum_{n=2}^{\infty}\sum_{m=-n}^{n} r_{nm}n(n+1)Y_n^m = \frac{\gamma}{a}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\left[n(n+1)-2\right]r_{nm}Y_n^m. \tag{12.21}$$

Finally:

$$\boxed{p_s^{(1)} = \frac{\gamma}{a}\sum_{n=2}^{\infty}\sum_{m=-n}^{n}(n-1)(n+2)r_{nm}Y_n^m.} \tag{12.22}$$

Stability. Summarizing the findings so far, we have shown that the small deviation from the spherical form, given by (12.8), leads to the deviation of the electrostatic and surface tensions pressures, as in (12.18) and (12.22), respectively. We note that the change of the air pressure within the bubble can be neglected for small deviations from the spherical form because the volume of the bubble remains constant to within $o(r_{nm}^2)$, as can be easily verified by integrating (12.8) over the full solid angle. The net force on the surface element has the same direction and intensity as the force produced by an external pressure of the magnitude:

$$\Delta p = p_0 - p + p_s - p_e = \cancel{(p_0 - p + p_s^{(0)} - p_e^{(0)})} + p_s^{(1)} - p_e^{(1)}. \tag{12.23}$$

Consider a deviation from a spherical shape which is proportional to a single spherical harmonic Y_n^m in (12.8). From (12.22), (12.18), and (12.23), we obtain:

$$\Delta r_{nm} = a r_{nm} Y_n^m(\theta,\phi), \qquad \Delta p_{nm} = (n-1)\left(\frac{\gamma}{a}(n+2) - \frac{1}{16\pi^2\varepsilon_0}\frac{q^2}{a^4}\right) r_{nm} Y_n^m(\theta,\phi)\,, \tag{12.24}$$

or alternatively:

$$\boxed{k_n = \frac{\Delta p_{nm}}{\Delta r_{nm}} = \frac{n-1}{a}\left(\frac{\gamma}{a}(n+2) - \frac{1}{16\pi^2\varepsilon_0}\frac{q^2}{a^4}\right)}, \qquad n \ge 2\,. \tag{12.25}$$

When $q = 0$ the restoring pressure is positive, indicating a stable equilibrium. As the charge q is gradually increased, the restoring pressure for all the perturbation modes diminishes. Other parameters being equal, the $n = 2$ mode exhibits the smallest restoring pressure. Consequently, the instability is first reached when $k_2 < 0$ in (12.25):

$$k_2 < 0 \quad \rightarrow \quad \frac{4\gamma}{a} - \frac{1}{16\pi^2\varepsilon_0}\frac{q^2}{a^4} < 0 \quad \rightarrow \quad p_s^{(0)} - p_e^{(0)} < 0 \quad \rightarrow \quad \boxed{p < p_0,} \tag{12.26}$$

here we have also used (12.6). The perturbation mode $n = 2$ is unstable when the pressure of the air within the bubble is less than that of the air outside, thus completing the proof.

Comment by Professor H.K. Moffatt on above solution :-

This problem is very Maxwellian! The above solution assumes a bubble of constant volume (but see alternative solution below). This means that there is no need to invoke an equation of state for the air inside the bubble, and to allow for the possibility that the bubble in fact expands as the electric charge on it increases. This seems a legitimate assumption, but it is an assumption that perhaps needs stating somewhere. Also, the dynamics of the air inside the bubble is also neglected; this is also legitimate in the stability calculation, although the reasons are not altogether obvious.

Alternative solution by Diego Sevilla

(where the assumption is that the bubble expands slowly, such that the bubble absorbs heat as it expands to keep its temperature constant)

1. Preliminaries

The air inside the bubble will be considered to be correctly described by the perfect gas state equation:-

$$pv = NkT \tag{12.27}$$

and all processes will be considered to be carried out sufficiently slowly to be isothermal, i.e. at a constant *T*. *N* (the number of moles of gas) is constant as no air escapes from the bubble and k is the gas constant.

We consider the surface tension constant (γ) independent of the soap's surface concentration; in this way, γ will not change when the bubble's surface is increased or decreased.

We shall study a bubble with the shape of an ellipsoid of revolution, whose deviation from the spherical form is given by the parameter *λ*, which measures the relative difference between the maximum radius of revolution and length of the ellipsoid along the axis of revolution. The volume, of this ellipsoid is:-

$$v = \frac{4}{3}\pi r^3 \left(1+\lambda\right) \tag{12.28}$$

From equations (12.27) and (12.28) one can obtain r as a function of p

$$r = \left(\frac{3NkT}{4\pi}\right)^{\frac{1}{3}} p^{-\frac{1}{3}} \left(1+\lambda\right)^{-\frac{1}{3}} \tag{12.29}$$

$$r = Kp^{-\frac{1}{3}} \left(1+\lambda\right)^{-\frac{1}{3}} \tag{12.30}$$

Where

$$K = \left(\frac{3NkT}{4\pi}\right)^{\frac{1}{3}} \tag{12.31}$$

In Section 2, the energy of the ellipsoidal bubble will be calculated. Then, in Section 3, the particular case of a spherical bubble will be considered, and the pressure of the air inside it will be calculated. Finally, in the last section, the conditions in which the spherical shape in no longer the most stable one are described.

1. 2 Energy calculation

We will call the bubble's total energy E, and this value can be expressed as the sum of three terms, which we will call volume energy (E_v), surface energy (E_s) and electrostatic energy (E_e). Volume energy is due to the free energy of the air inside the bubble (as T is constant), and changes when the bubble changes its volume. Surface energy is due to the surface tension of the film of soapy water, and it is proportional to the surface of the bubble. Finally, electrostatic energy depends on the charge and the electrical capacity of the bubble. So

$$E = E_v + E_s + E_e \tag{12.32}$$

2.1 Volume energy

The work carried out by the gas at pressure p, confined to a volume v and surrounded by gas at a constant p_0, when its volume is modified in dv, is

$$\delta W = \left(p - p_0\right) dv \tag{12.33}$$

If the temperature, T, of the system does not vary, then $\delta W = -dE_v$, being E_v the gas 'free energy' of Helmholtz. So

$$dE_v = -\left(p - p_0\right) dv \tag{12.34}$$

As T is constant, volume v is only a function of the pressure p. Using equation (12.27), one can write

$$dv = -\frac{NkT}{p^2} dp \tag{12.35}$$

Replacing (12.35) in (12.34)

$$dE_v = \frac{NkT}{p^2}\left(p - p_0\right)dp \quad (12.36)$$

By integrating, the volume energy is, ignoring the addition of a constant:-

$$E_v = NkT\ln\frac{p}{p_0} + NkT\frac{p_0}{p} \quad (12.37)$$

2.2 Surface energy

The energy due to the surface tension of a bubble is

$$E_s = 2\gamma S \quad (12.38)$$

being γ the coefficient of surface tension and *S* its surface. The surface of the ellipsoid of revolution is given in [1][1]

$$S = 2\pi b^2\{1 + \frac{a^2}{b\sqrt{a^2 - b^2}}\cos^{-1}(\frac{b}{a})\} \quad (12.39)$$

where *a* is the major semi-axis and *b* the minor semi-axes. We give *a* the value *(1+λ) r* and *b* the value *r*. So

$$S = 2\pi r^2\left\{1 + \frac{\left(1+\lambda\right)^2}{\sqrt{\lambda^2 + 2\lambda}}\cos^{-1}(\frac{1}{1+\lambda})\right\} \quad (12.40)$$

Considering equations (12.38), (12.40) and (12.30), the surface energy results in

$$E_s = 4\pi\gamma K^2 p^{-\frac{2}{3}} f\left(\lambda\right) \quad (12.41)$$

Where

$$f\left(\lambda\right) = \left(1+\lambda\right)^{-\frac{2}{3}}\left\{1 + \frac{\left(1+\lambda\right)^2}{\sqrt{\lambda^2 + 2\lambda}}\cos^{-1}(\frac{1}{1+\lambda})\right\} \quad (12.42)$$

[1] The formula will continue to be valid when *b>a* (flattened ellipsoid). By simple algebraical manipulations one can reach an equivalent formula $S = 2\pi b^2 + \frac{2\pi ba^2}{\sqrt{b^2 - a^2}}\ln\frac{b + \sqrt{b^2 - a^2}}{a}$

2.3 Electrostatic energy

A charged body's electrostatic energy is

$$E_e = \frac{Q^2}{2C} \qquad (12.43)$$

where Q is the electrical charge and C is the capacity. The capacity of a conducting surface with the shape of a ellipsoid of revolution is given in [2][2] where ε is a constant:-

$$C = 4\pi\varepsilon \frac{\sqrt{a^2-b^2}}{\ln\left(\dfrac{a+\sqrt{a^2-b^2}}{b}\right)} \qquad (12.44)$$

Once again, considering $a=(1+\lambda)\, r$ and $b=r$, it results that

$$C = 4\pi\varepsilon r \frac{\sqrt{\lambda^2+2\lambda}}{\ln\left(1+\lambda+\sqrt{\lambda^2+2\lambda}\right)} \qquad (12.45)$$

Considering equations (12.43), (12.45) and (12.30), one arrives at

$$E_e = \frac{Q^2}{8\pi\varepsilon K} p^{\frac{1}{3}} g(\lambda) \qquad (12.46)$$

being

$$g(\lambda) = (1+\lambda)^{\frac{1}{3}} \frac{\ln\left(1+\lambda+\sqrt{\lambda^2+2\lambda}\right)}{\sqrt{\lambda^2+2\lambda}} \qquad (12.47)$$

2.4 Total energy

Using (12.37) (12.41), and (12.46) in (12.32)

$$E = NkT\ln\frac{p}{p_0} + NkT\frac{p_0}{p} + 4\pi\gamma K^2 p^{-\frac{2}{3}} f(\lambda) + \frac{Q^2}{8\pi\varepsilon K} p^{\frac{1}{3}} g(\lambda) \qquad (12.48)$$

The first two terms of the Taylor's series of functions $f(\lambda)$ and $g(\lambda)$, given in (12.42) and (12.47) are (see Note below):-

[2] The formula is also valid for a flattened ellipsoid ($b>a$). By simple algebraical manipulations one can reach an equivalent formula $C = 4\pi\varepsilon \dfrac{\sqrt{b^2-a^2}}{\arccos\dfrac{a}{b}}$.

$$f(\lambda)=2+\frac{16}{45}\lambda^2+O(\lambda^3) \qquad (12.49)$$

$$g(\lambda)=1-\frac{4}{45}\lambda^2+O(\lambda^3) \qquad (12.50)$$

So, the energy of a partially deformed bubble (*λ<<1*) is

$$E=NkT\ln\frac{p}{p_0}+NkT\frac{p_0}{p}+8\pi\gamma K^2 p^{-\frac{2}{3}}+\frac{Q^2}{8\pi\varepsilon K}p^{\frac{1}{3}}+\frac{4}{45}\lambda^2\{16\pi\gamma K^2 p^{-\frac{2}{3}}-\frac{Q^2}{8\pi\varepsilon K}p^{\frac{1}{3}}\}+O(\lambda^3) \qquad (12.51)$$

3. The bubble's inner pressure

In a spherical bubble (*λ=0*), the energy is

$$E_{sph.}=NkT\{\ln\frac{p}{p_0}+\frac{p_0}{p}\}+8\pi\gamma K^2 p^{-\frac{2}{3}}+\frac{Q^2}{8\pi\varepsilon K}p^{\frac{1}{3}} \qquad (12.52)$$

If the bubble is in equilibrium, then

$$\frac{dE_{sph.}}{dp}=0$$

and based on this condition one obtains the following

$$p=p_0+\frac{1}{3}\frac{p^2}{NkT}\left\{16\pi\gamma K^2 p^{-\frac{5}{3}}-\frac{Q^2}{8\pi\varepsilon K}p^{-\frac{2}{3}}\right\} \qquad (12.53)$$

This equation defines *p* implicitly as a function of *N*, *T* and *Q*.

4. Stability of the bubble

According to equation (12.51), the energy of a slightly deformed bubble (*λ<<1*) is

$$E=E_{sph.}+V\lambda^2$$

Being

$$V=\frac{4}{45}\left\{16\pi\gamma K^2 p^{-\frac{2}{3}}-\frac{Q^2}{8\pi\varepsilon K}p^{\frac{1}{3}}\right\} \qquad (12.54)$$

One can easily see from equation (12.53) that the expresion between brackets is equivalent to

$$3NkT\left(1-\frac{p_0}{p}\right) \tag{12.55}$$

so that,

$$E = E_{sph.} + \lambda^2 \frac{12}{45} NkT\left(1-\frac{p_0}{p}\right) \tag{12.56}$$

In the last equation is presented how, when the inner pressure of the bubble *p* falls to a value less than the external pressure p_0, the spherical form is no longer the configuration with the least energy. In this way, when exposed to a minimal perturbation the bubble will tend to loose its shape and collapse. Another way of putting it is that, when $p=p_0$ there is no longer anything to sustain the bubble's shape –hence 'instability'.

Bibliography

1 V. V. Batiguin, I. N. Toptiguin. *Problemas de Electrodinámica y Teoría Especial de la Relatividad*. Ed. USSR, Moscow (1995), page 51.

2 V. V. Batiguin, I. N. Toptiguin. *Problemas de Electrodinámica y Teoría Especial de la Relatividad*. Ed. USSR, Moscow (1995), page 248.

Note:-

The expansion depends on

$$\{\frac{1}{(2x+x^2)^{\frac{1}{2}}}\}\cos^{-1}\{\frac{1}{(1+x)}\} = \{\frac{1}{(2x+x^2)^{\frac{1}{2}}}\}\sin^{-1}\{\frac{(2x+x^2)}{(1+x)^2}\}^{\frac{1}{2}} \tag{12.57}$$

and with

$$z = \frac{2x+x^2}{(1+x)^2} = 2x(1+x/2)(1+x)^{-2} \tag{12.58}$$

we have

$$\frac{\sin^{-1} z^{\frac{1}{2}}}{z^{\frac{1}{2}}} = 1 + \frac{z}{6} + \frac{3}{40}z^2 \tag{12.59}$$

$$\ln(1+x+\sqrt{x^2+2x}) = \cosh^{-1}(1+x) = \sinh^{-1}(\sqrt{x^2+2x}) \tag{12.60}$$

$$\frac{\sinh^{-1} z^{\frac{1}{2}}}{z^{\frac{1}{2}}} = 1 - \frac{z}{6} + \frac{3}{40}z^2 \tag{12.61}$$

Question 13. The resistance of a battery with its electrodes is *R*, and its electromotive force is constant. The circuit is completed by a fine wire of uniform section, whose resistance is a function of its temperature, which is supposed to be the same at all points of the wire. The wire is such that if no heat were generated in it, it would lose one per cent. of its excess of temperature over that of the air in a time *T*; and the electromotive force is such that if the wire were prevented from losing heat, its resistance would increase one per cent. in time τ. Show that if *r* is the resistance of the wire when the current is in equilibrium, the equilibrium will be unstable if (*R*-*r*)*T* is greater than (*R*+*r*).

Solution by David Forfar

Let θ_e be the equilibrium temperature of the wire, r_e be the equilibrium resistance of the wire and θ_a be the temperature of the air.

In equilibrium the current is $\dfrac{V}{R+r_e}$ and the rate of generation of heat is

$$\frac{V*r_e}{(R+r_e)^2} \tag{13.1}$$

this must equal the rate of heat loss in the wire which is

$$\frac{0.01*C}{T}(\theta_e-\theta_a) \tag{13.2}$$

where *C* is the thermal capacity of the wire. So the resistance and temperature in equilibrium are determined by $\dfrac{V*r_e}{(R+r_e)^2}=\dfrac{0.01*C*(\theta_e-\theta_a)}{T}$

In a small time Δt the increase in temperature of the wire is $\Delta\theta=\dfrac{.01*C*(\theta_e-\theta_a)}{T}\Delta t$ and the increase in resistance is $\Delta r=\dfrac{.01*r_e}{\tau}\Delta t$

The increase in the heat generated is the differential coefficient of (13.1) w.r.t. *r* times Δr which equals

$$\frac{V^2*(R-r_e)}{(R+r_e)^3}\Delta r=\frac{.01*V^2*r_e}{(R+r_e)^2}.\frac{(R-r_e)}{(R+r_e)}.\frac{\Delta t}{\tau} \tag{13.3}$$

The increase in heat loss is

$$\boxed{\frac{.01*C}{T}.\Delta\theta=\frac{V^2*r_e}{(R+r_e)^2}.\frac{\Delta\theta}{(\theta_e-\theta_a)}=\frac{.01*V^2*r_e}{(R+r_e)^2}.\frac{\Delta t}{T}} \tag{13.4}$$

If the rate of generation of heat is in excess of the capacity to lose heat, unstable equilibrium results.

This happens when

$$\boxed{(R-r_e).T>(R+r_e).\tau} \tag{13.5}$$

Question 14. It is proposed to construct a resistance coil, the percentage error of which shall be a minimum: the probable error *(take as meaning the standard deviation of the measurement of the resistance of the wire)* arising from imperfect connexion of the electrodes is *r*, and the defect of insulation is such that, independently of the wire, the conductivity between the electrodes is *C*, with a probable error (*take as meaning the standard deviation of C)* c: show that the best value for the resistance of the wire is such that if *x* is the actual resistance of the apparatus,

$$x^4c^2 = (1-Cx)^3(1+Cx)r^2$$

Solution by Jovan Jevtic

Let R be a random variable, representing the actual resistance of the wire and any imperfection arising from connection of the electrodes. We take the latter to be small relative to R. The actual resistance of the apparatus, x, is derived from the parallel connection of (a) the resistance R and (b) the conductivity between electrodes, C :

$$\frac{1}{x} = C + \frac{1}{R} \quad \text{or} \quad x = \frac{R}{RC+1} \tag{14.1}$$

If ΔR and ΔC are random variables representing small deviations from the true values of R and C and if $\boldsymbol{E}[\,]$ signifies "mathematical/statistical expectation" and given that ΔR and ΔC are assumed statistically independent of each other, we have:-

$$\boldsymbol{E}[\Delta R] = 0, \quad \boldsymbol{E}[\Delta C] = 0, \quad \boldsymbol{E}[\Delta R.\Delta C] = 0 \tag{14.2}$$

The change of x can be estimated as:

$$\Delta x = \frac{\partial x}{\partial R}\Delta R + \frac{\partial x}{\partial C}\Delta C \tag{14.3}$$

Consequently, we obtain, to first order, from (14.3) :

$$Variance(\Delta x) = \left(\frac{\partial x}{\partial R}\right)^2 Variance(\Delta R) + \left(\frac{\partial x}{\partial C}\right)^2 Variance(\Delta C) \tag{14.4}$$

If we now identify the variances of the random variables with the squares of *probable errors*, we have:-

$$Variance(\Delta R) = r^2, \qquad Variance(\Delta C) = c^2$$

Use (14.1) to evaluate the partial derivatives:

$$\frac{\partial x}{\partial R} = \frac{x^2}{R^2}, \qquad \frac{\partial x}{\partial C} = -x^2 \tag{14.5}$$

we obtain from (14.4) an expression for the square of the *relative error* (taken to be the variance of Δx divided by the square of x) to be minimized:

$$\frac{Variance(\Delta x)}{x^2} = \frac{x^2r^2}{R^4} + x^2c^2 \tag{14.6}$$

To minimize the *relative error* with respect to R, we set:

$$\frac{\partial}{\partial R}\left(\frac{x^2r^2}{R^4} + x^2c^2\right) = 0 \qquad \rightarrow \qquad R\left(r^2 + R^4c^2\right)\frac{\partial x}{\partial R} - 2xr^2 = 0 \tag{14.7}$$

which, with the help of the first expression in (14.5), gives the condition:

$$(r^2 + R^4c^2)x - 2Rr^2 = 0 \tag{14.8}$$

Finally, from (14.1) and substituting into (14.8), gives:

$$\left(1-Cx\right)^4 r^2 + x^4c^2 - 2\left(1-Cx\right)^3 r^2 = 0$$

$$\boxed{x^4c^2 = \left(1-Cx\right)^3 (1+Cx)r^2} \qquad (14.9)$$

thus completing the proof.

Question 15. The measured values of the sides of a certain triangle are $a,b,c,$ and the observed angles are found to exceed the angles calculated from $a,b,$ and c by $X,Y,$ and Z respectively: show that the most probable values of the sides are $a(1+x), b(1+y),$ and $c(1+z)$, where x,y,z are given by equations below,
where Δ is the area of the triangle, and the probable error *(take as meaning the standard deviation)* of a measurement of length is supposed to be l times that of a measurement of angle:-

$$\left[3a^4+(b^2-c^2)^2+8\Delta^2\frac{a^2}{l^2}\right]x+\left[c^4-(a^2+b^2)^2+8\Delta^2\right]y+\left[b^4-(c^2+a^2)^2+8\Delta^2\right]z$$
$$=2\Delta\left[2a^2X+(c^2-a^2-b^2)Y+(b^2-c^2-a^2)Z\right],$$

$$\left[c^4-(a^2+b^2)^2+8\Delta^2\right]x+\left[3b^4+(c^2-a^2)^2+8\Delta^2\frac{b^2}{l^2}\right]y+\left[a^4-(b^2+c^2)^2+8\Delta^2\right]z$$
$$=2\Delta\left[(c^2-a^2-b^2)X+2b^2Y+(a^2-b^2-c^2)Z\right],$$

$$\left[b^4-(c^2+a^2)^2+8\Delta^2\right]x+\left[a^4-(b^2+c^2)^2+8\Delta^2\right]y+\left[3c^4+(a^2-b^2)^2+8\Delta^2\frac{c^2}{l^2}\right]z$$
$$=2\Delta\left[(b^2-a^2-c^2)X+(a^2-b^2-c^2)Y+2c^2Z\right],$$

Comments:- x, y and z should be taken as very small, Δ is the area of the triangle calculated from a,b,c and the term 'probable error' should be taken as 'standard deviation' although the term 'standard deviation' was not coined by Karl Pearson (who sat the above examination in 1879) until 1893.

Solution by Jovan Jevtic

In the interest of clarity, we introduce the column vectors:

$$\vec{a}=\begin{bmatrix}a\\b\\c\end{bmatrix},\quad \vec{A}=\begin{bmatrix}A\\B\\C\end{bmatrix},\quad \vec{X}=\begin{bmatrix}X\\Y\\Z\end{bmatrix},\quad \vec{a}(\vec{x})=\begin{bmatrix}a(1+x)\\b(1+y)\\c(1+z)\end{bmatrix},\quad \vec{\alpha}(\vec{x})=\begin{bmatrix}\alpha(\vec{x})\\\beta(\vec{x})\\\chi(\vec{x})\end{bmatrix} \qquad ,(15.1)$$

Where *a,b,c* are the *measured* sides and *A,B,C* are the *measured* angles facing the sides a,b,c, respectively. Given $\vec{a}$, $\vec{x}$ defines a new triangle whose sides we denote by $\vec{a}(\vec{x})$ and whose angles (calculated from $\vec{a}(\vec{x})$) we denote by $\vec{\alpha}(\vec{x})$. The problem then reduces to the minimization of the error norm:

$$\frac{\left|\vec{\alpha}(\vec{x})-\vec{A}\right|^2}{\sigma_\alpha^2}+\frac{\left|\vec{a}(\vec{x})-\vec{a}\right|^2}{\sigma_a^2}=\text{minimum} \qquad (15.2)$$

with respect to $\vec{x}$. The error norm (15.2) is suitable for combining the results of two statistically independent measurements, $\vec{A}$ and $\vec{a}$, whose variances, σ_α^2 and σ_a^2 are different. From the problem statement: $\sigma_a = l\sigma_\alpha$ and $\vec{\alpha} = \vec{\alpha}(0) + \vec{X}$, so that (15.2) becomes:

$$\left|\vec{\alpha}(\vec{x}) - \vec{\alpha}(0) - \vec{X}\right|^2 + \frac{1}{l^2}\left|\vec{a}(\vec{x}) - \vec{a}\right|^2 = \min . \tag{15.3}$$

Small corrections. Consider first the angle $\alpha(\vec{x})$. From the cosine theorem:

$$\cos\alpha(\vec{x}) = \frac{b^2(1+y)^2 + c^2(1+z)^2 - a^2(1+x)^2}{2bc(1+y)(1+z)} . \tag{15.4}$$

Expanding into a Taylor series around $\vec{x} = 0$:

$$-\sin\alpha(0)\cdot\left(\alpha(\vec{x}) - \alpha(0)\right) = -\frac{a^2}{bc}x - \frac{c^2 - a^2 - b^2}{2bc}y - \frac{b^2 - c^2 - a^2}{2bc}z + o\left(\left|\vec{x}\right|^2\right). \tag{15.5}$$

After multiplying by $-2bc$ and noting that the calculated (from *a,b,c*) area equals $\Delta = \frac{1}{2}bc\sin\alpha(0)$:

$$4\Delta\left(\alpha(\vec{x}) - \alpha(0)\right) = 2a^2x + (c^2 - a^2 - b^2)y + (b^2 - c^2 - a^2)z + o(\left|\vec{x}\right|^2). \tag{15.6}$$

Similarly for $\beta(\vec{x})$ and $\gamma(\vec{x})$. For $|\vec{x}| \ll 1$, the results may be summarized as follows:

$$4\Delta\left(\vec{\alpha}(\vec{x}) - \vec{\alpha}(0)\right) = \mathbf{A}\cdot\vec{x}, \quad \text{where: } \boxed{\mathbf{A} = \begin{bmatrix} 2a^2 & c^2 - a^2 - b^2 & b^2 - c^2 - a^2 \\ c^2 - a^2 - b^2 & 2b^2 & a^2 - b^2 - c^2 \\ b^2 - a^2 - c^2 & a^2 - b^2 - c^2 & 2c^2 \end{bmatrix}.} \tag{15.7}$$

Minimization. In the light of (15.7), the error norm (15.3) reads:

$$\left|\mathbf{A}\cdot\vec{x} - 4\Delta\vec{X}\right|^2 + \frac{16\Delta^2}{l^2}\left|\vec{a}'(\vec{x}) - \vec{a}\right|^2 = \text{minimum}, \tag{15.8}$$

or, in terms of matrix and vector elements:

$$\sum_{i=1}^{3}\left(\sum_{j=1}^{3}A_{ij}x_j - 4\Delta X_i\right)^2 + \frac{16\Delta^2}{l^2}\sum_{i=1}^{3}\left(a_i x_i\right)^2 = \text{minimum}. \tag{15.9}$$

To minimize, we require $\partial / \partial x_k = 0$ for $k = 1,2,3$:

$$\begin{aligned}
&\sum_{i=1}^{3}\left(\sum_{j=1}^{3}A_{ij}x_j - 4\Delta X_i\right)A_{ik} + \frac{16\Delta^2}{l^2}a_k^2 x_k = 0, \\
&\sum_{j=1}^{3}\sum_{i=1}^{3}A_{ij}A_{ik}x_j - 4\Delta\sum_{i=1}^{3}A_{ik}X_i + \frac{16\Delta^2}{l^2}\sum_{j=1}^{3}a_k^2\delta_{kj}x_j = 0, \\
&\sum_{j=1}^{3}\sum_{i=1}^{3}\left(A_{ij}A_{ik} + \frac{16\Delta^2}{l^2}a_k^2\delta_{kj}\right)x_j = 4\Delta\sum_{j=1}^{3}A_{jk}X_j .
\end{aligned} \tag{15.10}$$

Dividing by 2 and noting from (15.7) that $A_{ij} = A_{ji}$, (15.10) reduces to:

$$\boxed{\left(\frac{1}{2}\mathbf{A}^2 + \frac{8\Delta^2}{l^2}\mathbf{D}\right)\cdot\vec{x} = 2\Delta\mathbf{A}\cdot\vec{X}, \quad \text{where: } \mathbf{D} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix}.} \tag{15.11}$$

We use (15.7) to find the square of the matrix $\mathbf{A}$. For example:

$$\left(\frac{1}{2}\mathbf{A}^2\right)_{11} = 3a^4 + b^4 - 2b^2c^2 + c^4, \quad \left(\frac{1}{2}\mathbf{A}^2\right)_{12} = a^2c^2 - a^2b^2 + b^2c^2 - \frac{3a^4 + 3b^4 - c^4}{2}. \quad (15.12)$$

Finally, we recall that the area of a triangle of sides a, b, c is given by:

$$\Delta = \frac{1}{4}\sqrt{a^2(b^2 + c^2 - a^2) + b^2(c^2 + a^2 - b^2) + c^2(a^2 + b^2 - c^2)} \quad (15.13)$$

which may be used in (15.12) to show that (15.11) is identical to the problem statement:

$$\left(\frac{1}{2}\mathbf{A}^2 + \frac{8\Delta^2}{l^2}\mathbf{D}\right)_{11} = 3a^4 + \left(b^2 - c^2\right)^2 + \frac{8\Delta^2}{l^2}a^2, \quad \left(\frac{1}{2}\mathbf{A}^2\right)_{12} = c^4 - (a^2 + b^2)^2 + 8\Delta^2, \quad (15.14)$$

and similarly for the remaining matrix elements, thus completing the proof.

Actual Marks awarded by Clerk Maxwell to the Candidates

Question	Marks	Sum Marks of Qn.	Attempts	Hill (5W)	Wallis (6W)	Pearson (3W)	Bell (8W)	Walker (2W)	Allen (SW)	Gunston (4W)	Lewis (24W)
1	10	40	4				10	10	10	10	
2a	20	40	2	20			20				
2b	40		0								
3a	50	200	4		50	50		50		50	
3b	25	0	0								
4	60	120	2					60		60	
5	50		0								
6	40	200	5			40	40	40	40		40
7	60	120	2			60		60			
8a	30	90	3	30		30			30		
8b	40		0								
9a	10		0								
9b	30	30	1	30							
10	50	150	3	50					50	50	
11a	20	100	5	20		20			20	20	20
11b	30	30	1	30							
12	30	120	4	30	30				30	30	
13	60	120	2		60				60		
14	50	50	1		50						
15	50	100	2		50	50					
Total Marks	755			210	**240**	**250**	70	220	**240**	220	60
Attempts				7	5	6	3	5	7	6	2

It is clear that the paper was too difficult for Lewis and Bell!