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The Changing Notation of Maxwell's Equations

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When, in his 1865 paper 'A Dynamical Theory of the Electromagnetic Field', Maxwell set out the equations now known as *Maxwell's Equations*, he used a notation in which the x, y and z components of each vector were assigned separate letters. For example,

Vector	x-component	y-component	z-component	Modern Symbol
Electric Field	P	Q	R	**E**
Magnetic Field	α	β	γ	**H**
Displacement	f	g	h	**D**
Free Current	p	q	r	**J**

Every vector equation was then written out as three simultaneous equations, one for each component. Noting that Maxwell wrote d where we would now write ∂, the modern equation $\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$ can be traced back to two equations that Maxwell labelled in this paper as (A) and (C). Taken together they give

$$d\gamma/dy - d\beta/dz = 4\pi(p + df/dt)$$
$$d\alpha/dz - d\gamma/dx = 4\pi(q + dg/dt)$$
$$d\beta/dx - d\alpha/dy = 4\pi(r + dh/dt)$$

The interest in these three *partial* differential equations, however, is that they represent one of Maxwell's greatest innovations. He realized that when the bound charges within an insulator were microscopically displaced under the influence of a time-varying electric field, a form of electric current would be created. This led him to conclude that the current in Ampère's original circuital law should be made up of not only the usual free current (represented by p, q, r) but also a *displacement current* represented by $df/dt, dg/dt, dh/dt$. In doing so, he created a new equation that not only guaranteed the conservation of electric charge but, taken together with Faraday's law of induction, gave rise to mutually supportive electric and magnetic fields, travelling at a finite speed in the form of waves. While several of these equations had appeared in an earlier article published by Maxwell in 1861-2, this was in connexion with an investigation of molecular vortices as a possible analogue for electromagnetic behaviour. In contrast, the equations in the 'Dynamical Theory' paper of 1865 expressed the behaviour of the electromagnetic field, a concept that originated with Faraday (1791-1867). These equations, and the subsequent developments discussed this article, therefore became the cornerstone of electrodynamics.

Based on some available experimental data that had nothing directly to do with the properties of light itself, Maxwell found that the speed of these waves was, within experimental error, equal to the known speed of light as determined by the French physicist Fizeau (1819-96). Thus, in 1865, Maxwell was able to affirm,

> *"…we have strong reason to conclude that light itself (including radiant heat and other radiation if any) is an electromagnetic disturbance in the form of waves…"*

This was just as Faraday had predicted some nineteen years earlier. Given the subsequent discovery of a whole spectrum of electromagnetic waves with an enormous range of wavelengths (Figure 2), these were prescient words indeed.

In his 1873 book, *A Treatise on Electricity and Magnetism,* Maxwell went on to express certain of his equations in the notation of quaternions, perhaps in deference to his friend and colleague, Tait (1831-1901), who was both a physicist and an expert on the subject. His key equations now appeared as in Figure 1. Quaternions were discovered by the Irish mathematician Hamilton (1805-65); they have four components, one of which is a scalar while the other three form a vector. In quaternion language, ∇ represents the vector derivative $\partial_x\mathbf{i}+\partial_x\mathbf{j}+\partial_z\mathbf{k}$, while $\mathfrak{H}$ represents the vector $H_x\mathbf{i}+H_y\mathbf{j}+H_z\mathbf{k}$ (*i.e.* $\mathbf{H}$ in the table above). The product of ∇ and $\mathfrak{H}$ is then written as $\nabla\mathfrak{H}$ and, by applying Hamilton's original rule $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$ and preserving the order of products, $\nabla\mathfrak{H}$ may be evaluated as

$$\nabla\mathfrak{H} = \underbrace{-\left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right)}_{scalar\ \ part} + \underbrace{\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\mathbf{k}}_{vector\ \ part}$$

This equation was formally written as $\nabla\mathfrak{H}=S.\nabla\mathfrak{H}+V.\nabla\mathfrak{H}$, in which $S.\nabla\mathfrak{H}$ corresponds to $-\nabla\cdot\mathbf{H}$ while $V.\nabla\mathfrak{H}$ corresponds to $\nabla\times\mathbf{H}$. Today, however, most people would agree that Maxwell's equations appear simpler when written in the vector notation due independently to Heaviside (1850-1925) and Gibbs (1839-1903). In Heaviside's formalism $V.\nabla\mathfrak{H}=\mathfrak{K}+\dot{\mathfrak{D}}$ became $\operatorname{curl}\mathbf{H}=\mathbf{J}+\partial\mathbf{D}/\partial t$, whereas Gibbs introduced the more familiar notation $\nabla\cdot$ and $\nabla\times$ that sometime later led to $\nabla\times\mathbf{H}=\mathbf{J}+\partial\mathbf{D}/\partial t$. Apart from such notational differences, their vector representations were identical.

In addition to introducing vector notation, Heaviside made other important changes to Maxwell's quaternion equations. First, taking the equation labelled in the *Treatise* as (*B*), he removed the expression relating to the velocity of a moving conductor and, by taking the *curl* of the remaining equation, he eliminated the scalar potential. He was then able to replace the curl of **A** (the vector potential) with **B** (the magnetic induction) to obtain $\operatorname{curl}\mathbf{E}=-\partial\mathbf{B}/\partial t$. It is interesting to note that the term *curl* was suggested first by Maxwell. Finally, Heaviside eliminated the vector potential from the equation that Maxwell labelled as equation (*A*) by taking its *divergence*, giving $\operatorname{div}\mathbf{B}=0$. Heaviside preferred *divergence* to Maxwell's opposite idea of *convergence.*

Lorentz (1853-1938) then made a further contribution by bringing in the microscopic origin of electric charge. In doing so, he expressed the displacement **D** as $\varepsilon_0\mathbf{E}+\mathbf{P}$, where **P** is the *dielectric polarisation* that is associated with the displacement of local charge, which is separate from the *vacuum polarisation* $\varepsilon_0\mathbf{E}$ in which there is no *local* displacement of charge. Taken together, these changes gave Maxwell's equations the foundation and form recognised by today's scientists and engineers.

The equations on the plinth of the statue of Maxwell in Edinburgh (Figure 3) are,

$$\nabla\cdot\mathbf{D}=\rho \qquad \nabla\cdot\mathbf{B}=0 \qquad \nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t \qquad \nabla\times\mathbf{H}=\partial\mathbf{D}/\partial t+\mathbf{J}$$

In 1887, Hertz (1857-94) demonstrated electromagnetic signals being transmitted and received across his laboratory; they travelled at a finite speed, were reflected from surfaces and caused interference patterns, just like light waves. This vindicated experimentally Faraday's and Maxwell's unique insights and today the achievements of Faraday, Maxwell and Hertz stand amongst the greatest in physics. Unfortunately, Faraday and Maxwell did not live to share in the triumph that resulted from Hertz's discovery. Hertz went on, in 1890, to publish a clarified version of Maxwell's equations. Although these were mathematically equivalent to Heaviside's version, Hertz did not adopt the new vector notation and simply wrote out his equations the long way.

Einstein's admiration for Maxwell has been recorded in an earlier newsletter which quoted Einstein as saying *"…I stood on the shoulders of Maxwell."* Indeed, one of the cornerstones of Einstein's 1905 paper on special relativity is that measurements made by observers with different uniform velocities are related by a Lorentz transformation, and not by a simple Galilean one. While the Lorentz transformation may at first seem counter-intuitive, it guarantees the constancy (invariance) of the speed of light in free space, that is to say it always has the same value for *any* observer no matter what their relative speed. Not only do Maxwell's equations lead to this same conclusion, they are fully compatible with Einstein's special relativity just as they stand. This was one of the key observations that guided Einstein to his remarkable conclusions.

By using Minkowski's 4-dimensional spacetime (three dimensions for space and one for time) and the formalism of tensors, Einstein later found that Maxwell's equations in free space may expressed in a form that is consistent with general relativity

$$\partial F_{\rho\sigma} / \partial x_\tau + \partial F_{\sigma\tau} / \partial x_\rho + \partial F_{\tau\rho} / \partial x_\sigma = 0 \qquad \partial F^{\mu\nu} / \partial x_\nu = J^\mu$$

The components of the 4×4 antisymmetric tensor $\boldsymbol{F}$ express the components of both the electric field *and* magnetic fields as seen by an arbitrary observer. However, while the components of $\boldsymbol{x}$, $\boldsymbol{F}$ and $\boldsymbol{J}$ may change from one observer to another, they will always obey these same tensor equations. Such equations are said to be *covariant*, a hallmark of all of Nature's fundamental physical equations.

With advances in mathematical notation have come increasingly succinct ways of writing Maxwell's equations. This may involve the use of differential forms (pioneered by the French mathematician, Cartan 1869-1951), geometric algebra (revived by Hestenes, 1933-) or biquaternions (Graves 1806-1870 and Cayley 1821-1895). For example, in notation of biquaternions, Maxwell's equations in free space become

$$\left(\nabla - \frac{i}{c}\frac{\partial}{\partial t} \right)\left(\mathbf{E} + i\,c\mathbf{B} \right) = -\frac{\rho}{\varepsilon_0} + i\,Z_0\mathbf{J}$$

while in the notation of differential forms they are

$$dF = 0 \qquad d{*}F = J$$

Finally, in the notation of geometric algebra, the equations take the even more concise form,

$$\nabla F = J$$

But, although mathematical techniques have developed and the notation has become more succinct, the underlying physical behaviour of electromagnetic fields is essentially just as Maxwell expressed it in his equations of 1865. ■

	Quaternion Equation	Modern Translation	Heaviside's Equation
(A)	$\mathfrak{B} = V.\nabla\mathfrak{A}$	$\mathbf{B} = \nabla \times \mathbf{A}$	$\mathrm{div}\,\mathbf{B} = 0$
(B)	$\mathfrak{E} = V.\mathfrak{G}\mathfrak{B} - \dot{\mathfrak{A}} - \nabla\psi$	$\mathbf{E} = \mathbf{v} \times \mathbf{B} - \partial\mathbf{A}/\partial t - \nabla\phi$	$\mathrm{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t$
(E)	$4\pi\mathfrak{C} = V.\nabla\mathfrak{H}$	$\mathbf{J}^{total} = \nabla \times \mathbf{H}$	$\mathbf{J}^{total} = \mathrm{curl}\,\mathbf{H}$
(H)	$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}}$	$\mathbf{J}^{total} = \mathbf{J} + \partial\mathbf{D}/\partial t$	$\mathbf{J}^{total} = \mathbf{J} + \partial\mathbf{D}/\partial t$
(J)	$e = -S.\nabla\mathfrak{D}$	$\rho = \nabla \cdot \mathbf{D}$	$\rho = \mathrm{div}\,\mathbf{D}$

Figure 1: The key electromagnetic equations from Maxwell's *Treatise on Electricity and Magnetism* of 1873. On each row, his quaternion form of the equation is shown, then its modern translation. This is followed by Heaviside's version, effectively as published in the *Electrician* of 1885. Note that $V.\mathbf{ab} \equiv \mathbf{a} \times \mathbf{b}$, $S.\mathbf{ab} \equiv -\mathbf{a} \cdot \mathbf{b}$, $\mathfrak{A}$ reads as $\mathbf{A}$, while $\mathfrak{G}$ (a velocity) has been replaced, in the modern translation, by the more familiar $\mathbf{v}$.

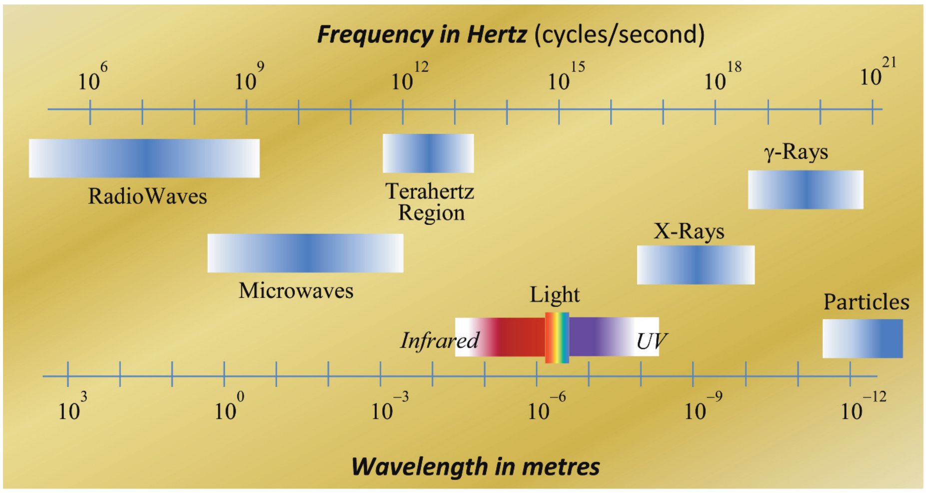

Figure 2: Although the spectrum of electromagnetic waves has no actual limits, the diagram shows the best known part of it, beginning with wavelengths of 1km, where radio broadcast starts, all the way to the shortest wavelengths that are associated with the behaviour of physical particles.

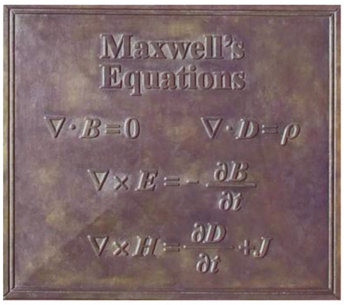

Figure 3: Maxwell's equations as they are depicted on his commemorative statue in George Street, Edinburgh. The statue is just a short walk from Maxwell's place of birth at 14 India Street, now the home of the James Clerk Maxwell Foundation (courtesy of the Royal Society of Edinburgh and Sir Michael Atiyah).

This article is based on: J. W. Arthur, "The Evolution of Maxwell's Equations from 1862 to the Present Day", to appear in *IEEE Antennas and Propagation Magazine*, http://www.ieeeaps.org/magazine.html

Advance notice of a new book about the contributions to physics of James Clerk Maxwell

Flood R. et al., (publication expected 2013), James Clerk Maxwell (1831-1879), edited by Flood R., McCartney M. and Whitaker A., Oxford University Press.