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Maxwell's Articles on Structural Mechanics

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'Structural mechanics' is the mathematical logic and procedures used to seek to ensure that structures, that support a load, perform satisfactorily. Maxwell's contributions to structural mechanics were in three main areas namely:

(1) stresses and deformations of elastic solids,

(2) reciprocal diagrams for determining forces in determinate frames (i.e. just enough members to provide a stiff frame) and,

(3) methods for determining forces and displacements in indeterminate frames (i.e. more members in the frame than the minimum needed for stiffness).

A Historical Perspective of Structural Mechanics

Starting with an unsuccessful attempt by Galileo in 1638, for two hundred years the mathematicians of Europe attempted to solve, unsuccessfully, the problem of predicting the strength of a beam. Thus, the required specification of the size of beams could only be done on the basis of past experience and by trial and error. Bridge failures were common.

The great bridge builder, Thomas Telford (1757–1834), was rather disappointed that the mathematicians up to his day had not been able to provide him, as late as the early eighteen hundreds, with much help for the design of his masterpieces, such as the Craigellachie Bridge (designed in 1812, see Fig. 1) and the Menai Suspension Bridge (designed in 1819, see Fig. 2).

Fig. 1: Telford's Craigellachie Bridge

Fig. 2 : Telford's Menai Suspension Bridge

However, in 1826, the Frenchman Claude-Louis Navier[1] (1785 – 1836) published a volume of his 'Lecons' that included a method for reliably predicting the behaviour of a beam. After this, methods of structural mechanics burgeoned and the ability to predict both internal forces and the displacement of structures under load significantly improved. As a result, the risk of failures caused by design errors dropped to a low level. However, the analysis of indeterminate structures was still limited by the effort required to solve simultaneous equations.

When computers and 'finite element methods' were introduced in the 1950s and 1960s there was a major increase in the potential to provide solutions to structural mechanics problems. This led to another step change in the use of structural mechanics.

'The Equilibrium of Elastic Solids'

This was the title of the first paper by Maxwell on structural mechanics. It was read to the Royal Society of Edinburgh in 1850 by Professor Philip Kelland, the Professor of Mathematics at Edinburgh University. It shows astonishing maturity for a person who was not yet nineteen years of age. Maxwell refers in it to a wide range of sources including the Frenchmen, Navier (1785 – 1836), Cauchy (1789 – 1857), Poisson (1781 – 1840) and Clapeyron (1799 – 1864) and the Irishman Lord Stokes (1819 – 1903). Maxwell seems to have been able to read the foreign papers in the language in which they were written. The paper started by deriving (for a solid body) the differential equations of elasticity, based on two axioms.

The equations which Maxwell derived were the same as those of Navier and Cauchy, although they had used a different approach to their derivation. Maxwell then proceeded to apply the equations to a number of cases. The three cases of most interest to current structural engineers are :

1) the stiffness of a hollow cylinder (fixed at one end) and twisted at the other end by a couple. Maxwell derives an expression for the degree of rotation of such a 'twisted' cylinder. As Maxwell did not make reference to other sources, his expression may be the first time it was published. Maxwell's expression is in common use to-day,

2) the stiffness of a rectangular beam under bending. Maxwell derived an expression which agrees with Navier,

3) the stiffness of a beam taking account of bending and shear deformation. He makes no reference to other sources for the shear deformation component and therefore this may be original (but there is an error in this part of the derivation which has subsequently been corrected).

1 famous for the Navier-Stokes equation in fluid mechanics.

The paper also reported on the theoretical studies and experiments that Maxwell had made on the phenomenon of photo-elasticity (where polarised light shows up the strains in certain types of glass or crystal) which had been discovered by Sir David Brewster (1781 – 1868). The results of these experiments were used to seek to define the elastic constants in the differential equations.

Determinancy or indeterminancy/redundancy of frames

Fig 3 shows diagrams that represent models of pin-connected frames i.e. frames made up of members connected such that their ends are free to rotate independently.

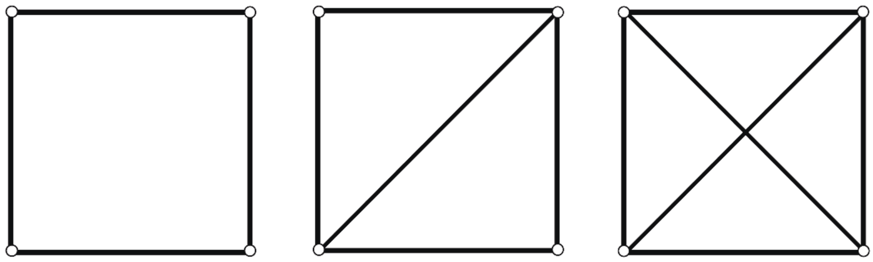
Fig. 3a shows a rectangular frame without a diagonal bracing member. This frame would collapse under any loading in its plane. Adding a diagonal member (Fig. 3b) provides the frame with just enough members to make it 'stiff' and able to support a load (this is a 'determinate' frame). In Fig. 3c, there is one member that is extra to the requirements for being determinate (this is an 'indeterminate' frame with one 'redundant' member).

Maxwell established (in his second paper on structural mechanics – see below) that, in two dimensions, a frame is determinate if:

(number of members) – (2 x number of points) + 3 = 0

For example, for the frame in Fig. 3b, the number of members is 5 and the number of points is 4 so the criterion for determinacy is satisfied. For Fig. 3c, the number of members is 6 and the number of points is 4. The criterion is not satisfied and the degree of indeterminacy of the frame is 1.

Fig. 3 Determinate or indeterminate frames

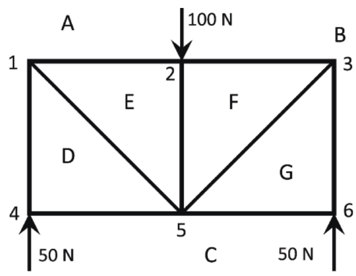


(a) Frame *(b) Determinate or 'stiff'* *(c) Indeterminate frame with redundant member*

'On Reciprocal Figures and Diagrams of Forces'

This was the title of the second paper on structural mechanics published by Maxwell in 1864. It explains the theoretical background to the use of reciprocal diagrams that are used to calculate the internal forces in a 'pin-jointed' determinate frame of members i.e. of the type shown in Figure 3(b). Maxwell 'mathematised' the two diagrams of Macquorn Rankine (1821 – 72) – one showing a diagram of the frame and the other showing the diagram of forces in the members of the frame. Maxwell appreciated that they were reciprocal diagrams in the geometrical sense that, in the two diagrams, the number of lines was equal, the lines were parallel and the lines meeting in a point in the first diagram translated into polygons in the second diagram and vice versa. Thus the lengths of the lines in the diagram of forces are (to scale) the forces that act along the members of the frame in order to secure the frame's equilibrium. Fig. 4 demonstrates the method.

Fig. 4: Construction of a reciprocal diagram for a frame



(a) Frame diagram of a structure supporting a 100 Newton load

In (a) the numbers 1–6 are 'point' numbers. A 'member' (Maxwell describes this as a 'piece') connects two points. A member is denoted by the points to which it is connected and the forces in the members are defined by the pair of letters on either side. For example, the force in member 1–2 is AE.

(b) is the reciprocal diagram to (a) and is a composite of the force polygons at each point in the frame. C, D and G are at the same position because there are no forces in members 4–5 and 5–6. Thus the force 'polygon' for point 4 is the pair of superimposed lines CA and AD.

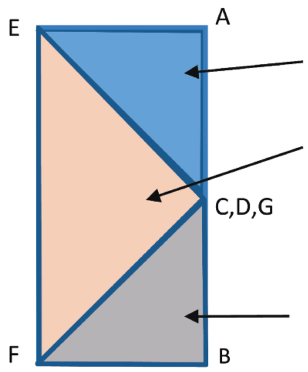


Fig. 4(b), Triangle of forces for point 1, so point 1 has translated into a 3-sided polygon

Triangle of forces for point 5, so point 5 has translated into a 3-sided polygon. This is not a pentagon because members 4 – 5 and 5 – 6 have zero force for this loading

Triangle of forces for point 3, so point 3 has translated into a 3-sided polygon

Member	Force	Value of force (N)
4-1	AD	-50
4-5	CD	0
1-2	AE	-50
1-5	DE	$50\sqrt{2}$
2-5	EF	-100
2-3	BF	-50
3-5	FG	$50\sqrt{2}$
3-6	BG	-50
3-6	CG	0

The sign convention for the Forces is: tension +ve.

For a practical example, Fig. 5a shows a triangulated pin-jointed bridge frame together with seventeen external loads. The length of the lines in the corresponding reciprocal diagram, Fig. 5b, shows the magnitude of the forces acting along the members. It will be seen from Fig. 5b that the greatest force is TH. Fig 5c shows a similar modern bridge structure.

Fig. 5(a) and Fig. 5(b): Diagrams taken from the entry by Fleeming Jenkin in the 1878 (9th) Encyclopedia Britannica of which Maxwell was joint scientific editor

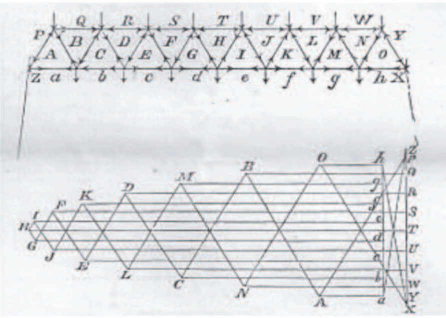


Fig. 5(c): A modern railway bridge consisting of triangulated frames

A Mr. Taylor[2] and Professor Rankine had used reciprocal diagrams before Maxwell but it was Maxwell who gave them their proper mathematical formulation and geometric context. For example, Robert Bow[3] wrote:

"The importance to which this method (use of reciprocal diagrams) has obtained is almost altogether due to Professor Clerk Maxwell who has shown how surprisingly general its application is and who has placed it in quite a new aspect by his discovery or detection of those diagrams of forces which bear a reciprocal relationship to the relative framed structure."

'Calculation of Equilibrium and Stiffness of Frames'

In this paper, published also in 1864, Maxwell develops a method for solving indeterminate pin-jointed frames. To achieve this, it is necessary to take account not only of the equilibrium of the forces but also to consider the extensibility of the members of the frame. The paper is remarkable for the number of important concepts introduced in its six pages. The concepts are:

The degree of indeterminancy of a frame

Maxwell establishes the criterion for establishing the degree of determinacy of a three dimensional frame, as he had done (in his earlier paper) for a two dimensional frame.

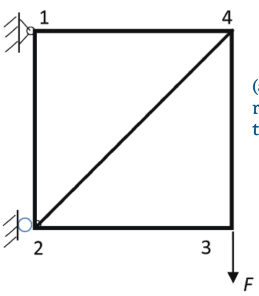
The principle of contragredience

Maxwell proved, by using the principle of conservation of energy, that if, in a structure, a tension of one unit between B and C results in a tension, p say, in member A then correspondingly, an extension in member A of one unit of length, will bring B and C nearer together by an amount of p units of length. This principle is nowadays known as the *'Principle of Contragredience'*. This principle, extended to the use of vectors rather than only to scalar quantities, is now used as a standard technique in the development of theory for solving indeterminate structures.

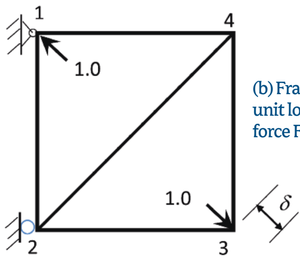
The unit load method

Fig. 6a shows an external downward force F applied to a determinate frame. The resulting displacement (δ) of the frame (due to extensibility of the members) in the diagonal direction at a point 3 is to be determined.

Fig. 6: The Unit Load Method



(a) Frame supported at 1 with roller at 2 (allowing vertical movement) taking a vertical load F at point 3



(b) Frame as in (a) with a 'virtual' unit load on the diagonal and force F removed, as shown

Reciprocal diagrams or other means are used to calculate the forces in the members of the frames of Fig 6(a) and (b). The effect of changes in geometry of the structure due to the changes in lengths of the members is ignored in the calculations.

When $F = 1$, the force along the diagonal 2–4, for example, is say, p. For a load F, the force will be $F*p$ (Fig. 6a).

In Fig 6(b), load F is removed and the force in member 2–4 due to the unit diagonal load along 1–3 is defined as, say, r.

The extension of member 2–4 due to F is $F*e*p$ where e is the extensibility of member 2–4. By using the principle of contragredience (see above), the diagonal movement in the line of δ at point 3 due to F is equal to $F*e*p*r$. This calculation is repeated for all the members of the frame and the results added to obtain the total movement in the line of δ due to F as $F\sum epr$.

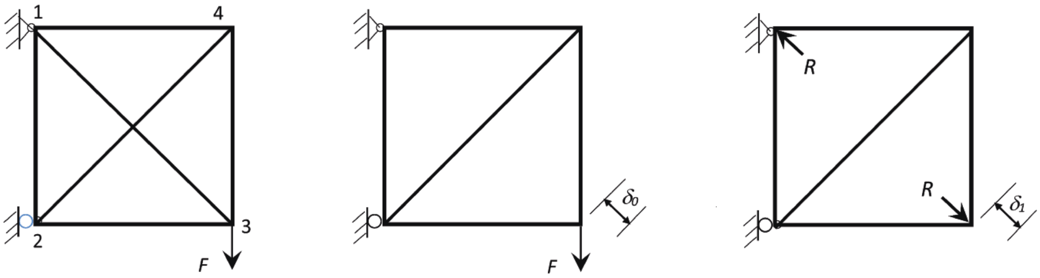
2 In 1869, in a memoir to the Royal Society of Edinburgh, Professor Fleeming Jenkin (Professor of Engineering at Edinburgh University) wrote *"…the authors attention was drawn to the method (of reciprocal diagrams) by the circumstance that it was independently discovered by a practical draughtsman, Mr. Taylor, working in the office of a well-known contractor, Mr. J.B. Cochrane."*

3 Robert Bow (1832 – 1908) was a Scottish engineer known for Bow's Notation for reciprocal figures.

The flexibility method

Maxwell had now set up the tools to determine the forces in indeterminate frames.

Figure 7: The Flexibility Method



(a) Frame with applied load

(b) Frame with redundant diagonal removed

(c) Frame with force *R* in the diagonal direction and force F removed

Fig. 7a shows the same frame as Fig. 6a, with the exception of an extra diagonal, 1–3, making it indeterminate. The force, *R* say, in the diagonal member, 1–3, is to be determined.

Using the 'unit load method' (see above), the displacement of point 3 can be calculated. The change in length on the diagonal, 1–3:

- due to force F (Fig. 7b) is $\delta_0 = F\sum epr$
- due to R (Fig. 7c), is $\delta_1 = R\sum er^2$

When both forces F and R act together, then the extension $R\rho$ of the diagonal member 1–3 must be $-(\delta_0 + \delta_1)$

Thus the force, R, in the diagonal member, 1–3, is easily found by solving the 'compatibility equation', $R\rho + \delta_0 + \delta_1 = R\rho + F\sum epr + R\sum er^2 = 0$ where ρ is the extensibility of the diagonal member 1–3.

If there is more than one degree of redundancy then simultaneous equations need to be solved. In this regard Neville Shute, before he became a novelist, was the aeronautical engineer responsible for the structural calculations of the frames of the R100 airship in the 1920s. He recorded that it would take two weeks to solve the resulting set of simultaneous equations for one frame. (It is most likely that he used the flexibility method for this calculation.) The full (non-linear) analysis of the frame would take two to three weeks. With modern computers the solution of such equations can be done in fractions of a second.

Maxwell Reciprocal Theorem

In this paper, Maxwell makes the statement:

"The extension in BC due to unity of tension along DE is always equal to the extension in DE due to unity of tension in BC." (where *BC* and *DE* are frame members).

This is known as the 'Clerk Maxwell Reciprocal Theorem'. It has been used in a number of ways in structural mechanics. The symmetry of elastic force-deformation relationships follows directly from it.

The Maxwell-von Mises criterion for plastic material failure

In a letter to William Thomson (Lord Kelvin) in 1854, Maxwell stated that he had never seen any investigation of the question *"Given the mechanical strain in three directions on an element, when will it give way?"*. He wrote that he had *"strong reasons for believing"* that when the distortional strain energy of the element reached a certain critical value then the material would reach its plastic limit and give way. This has proved to be an important insight. What is known as the 'von Mises criterion' for plastic failure of a material is based on this principle and is now in common use.

Maxwell's legacy

Maxwell identified the mathematical context for reciprocal diagrams for the forces in pin-jointed frames under load. He also devised a method for calculating the displacements due to the extensibility of the members of such frames and used this to solve the problem of calculating the forces in frames that are indeterminate. While he did develop many of these methods independently, he may not have always been the first (for example Rankine and Taylor had used reciprocal diagrams) and was not the last.

It is pertinent to ask the question as to why Maxwell spent time on structural mechanics while he was developing the theory of electromagnetism and other deep topics in physics. As well as being a superb theorist, Maxwell was a practical man and must have realised that there are few areas in applied science that are more directly related to the benefit of mankind than structural mechanics.

That modern buildings, bridges, aircraft are very unlikely to fail due to design errors is largely as a result of advances in structural mechanics in the 19th and 20th centuries.

Maxwell turned the searchlight of his genius onto this subject and made important contributions to its development.

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